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Optimal hedging by the Martingale method in complete and incomplete markets

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OPTIMAL HEDGING BY THE MARTINGALE METHOD IN COMPLETE AND INCOMPLETE MARKETS

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Abstract

This paper applies the Martingale method proposed by Cox and Huang to find optimal wealth and optimal hedging strategies in the Black-Scholes-Vasicek economy. It starts with a review of the general solution of optimal wealth by the Martingale method, and then apply it to a power utility function. We investigate optimal hedging under three benchmark scenarios including a cash, a stock, and an inflation-indexed bond benchmark. We analyse optimal portfolio choices when inflation-indexed bonds are traded in complete markets and not traded in incomplete markets. The outcomes suggest that optimal wealth and hedging highly depend on the volatilities of benchmarks and the degree of risk aversion of investors.

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Chapter 1

Introduction

Optimal hedging problem studies how agents should allocate wealth among different financial assets in order to maximise their expected utilities at a given time. Pension funds routinely adjust asset positions to meet their financial objectives. One of the common objectives of pension funds is outperforming predetermined benchmarks. Insurance companies often set up benchmarks to monitor the performance of pension funds over the life cycle, and optimal asset allocation strategies highly depend on the characteristics of benchmarks. In general, long-term pension funds often confront with risks that cannot be traded, such as inflation risks, mortality risks, etc. Therefore, pension funds ought to take into account these unhedged risks to determine optimal hedging strategies in incomplete markets.

The classic portfolio optimisation problem was first formulated by Merton (1969), where he provided explicit solutions of optimal hedging strategies for investors to maximise their expected utilities. In the financial setting, risk-free money market accounts have constant interest rates, and risky stocks follow geometric Brownian motions. Merton (1969) used the dynamic programming approach to transform the stochastic investment problem into a Hamilton-Jacobi-Bellman (HJB) equation. After solving the HJB equation, explicit solutions for optimal asset allocation strategies can be obtained. This approach works well for hyperbolic average risk aversion (HARA) utility where partial differential equations (PDE) are linear. However, numerical solutions are difficult to obtain in the case of non-linear PDEs.

The Martingale method introduced by Harrison & Kreps (1979) is an alternative approach to find optimal portfolios. Further development of this approach was achieved by Pliska (1986), Karatzas et al. (1987), and Cox & Huang (1989), where closed-form solutions of optimal wealth and hedging strategies were derived without solving PDEs. These papers considered the complete market setting, where a unique optimal self-financing strategy exists, and all payoffs of portfolios can be perfectly replicated. Cox & Huang (1989) used the Lagrangian method to transform a utility optimisation problem with budget constrains into an unconstrained optimisation problem. Optimal solutions can be obtained after solving the first-order conditions of the unconstrained optimisation.

In incomplete markets there exist risks that cannot be traded. In this case, the arbitrage-free condition is not satisfied, and equivalent martingale measures are not unique. Therefore, fair prices cannot be derived without constructing perfect replicating portfolios. A super-hedging strategy was proposed by El Karoui & Quenez (1995), Jouini & Kallal (1995), and Karatzas (1997) to determine optimal hedging from super-hedging prices in incomplete markets. However, the super-hedging strategy often leads to high costs and low opportunity to gain profits. Föllmer & Leukert (1999) constructed an alternative approach called quantile hedging strategy where the cost of hedging is minimised, and the probability of a successful hedging strategy is maximised.

Modern portfolio analysis and asset allocation require specific financial settings. Merton (1969) derived the optimal solution based on the classic Black-Scholes (1973) economy where interest rates are assumed to be constant. Many studies extended the Black-Scholes (1973) model with stochastic interest rates following the Ornstein-Uhlenbeck process. One of the main types of stochastic short rates is the Vasicek (1977) model, see Omberg (1999), Brennan & Xia (2002), Eksi (2007), and Angelini & Herzel (2014). The Black-Scholes-Hull-White economy is another common type of economic scenario where shorts rates follow the Hull-White (1993) model, and relevant studies are Renault & Touzi (1996), Biagini et al. (2000), Yang et al. (2010), and Goutte (2013).

Optimal hedging strategies highly depend on the degree of risk aversion of investors, and utility functions capture risk preferences. In general, explicit solutions can be derived under HARA utility. Power utility is a type of HARA with constant relative risk aversion, and Brennan & Xia (2002), Kallsen et al. (2014), and Adam-Müller (2000) discussed optimal portfolio choices under power utilities. Other common HARA utility functions for portfolio optimisation are exponential utility (cf. Mania & Tevzadze (2008), Browne (1995)) and log-utility (cf. Pang (2006), Matsumoto (2006)). Expected shortfall can also be incorporated in a utility function. Föllmer & Leukert (2000), Pochart & Bouchaud (2004), and Gabih et al. (2005) derived optimal hedging strategies where the utility is maximised, and the expected shortfall is minimised.

This paper focuses on applying the Martingale method proposed by Cox & Huang (1989) to find optimal wealth and optimal hedging strategies in the Black-Scholes-

Vasicek economy where the stochastic interest rate follows the Vasicek (1977) model. It investigates optimal hedging under three benchmark scenarios including a cash, a stock, and an inflation-indexed bond benchmark. Inflation risk is the non-traded risk in the financial market. Two cases are discussed when inflation-indexed bonds are available in complete markets and not available in incomplete markets.

The rest of this paper is structured as follows. Chapter 2 introduces the Black-Scholes-Vasicek model and the concept of risk-neutral pricing. Chapter 3 explains the dynamic optimisation problem. We provide a general solution of finding optimal wealth and optimal hedging strategies by the martingale method, and then apply the method to a power utility function. In chapter 4, we derive optimal wealth under a cash, a stock, and an inflation-indexed bond benchmark respectively. Chapter 5 analyses the way to compute optimal hedging strategies in both complete and incomplete markets, and chapter 6 concludes.

Chapter 2

Financial Market and Risk Neutral Valuation

This chapter introduces a financial market model in the Black-Scholes-Vasicek economy. It explains the concept of no-arbitrage and the risk-neutral valuation approach to price any asset. Further, it presents the portfolio wealth where we include three types of assets including stocks, money market accounts, and zero-coupon bonds.

2.1 Black-Scholes-Vasicek Model

In the general Black-Scholes (1973) economy, expected returns and volatilities of asset price processes are assumed to be time-independent. The Black-Scholes-Vasicek model is an extension of the Black-Scholes model with a stochastic interest rate that follows the Vasicek (1977) model.

Define a time $T \in \mathbb{R}_+$, and a time span $[0, T]$. Let us consider a financial market under a probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, \mathbb{P})$, where Ω is a sample space, $\{\mathscr{F}_t\}_{t\in[0,T]}$ is the filtration, and $\mathbb{P}$ is a probability measure. Additionally, let $W = \{W_{t,S}^{\mathbb{P}}, W_{t,r}^{\mathbb{P}}\}$ be two correlated standard Brownian motions defined on this probability space. These two Brownian motions are the main sources of randomness in the financial market.

The market consists of both risky and riskless assets. Let B_t be the price process of the risk-free money market account, and S_t be the process of the stock that is assumed to be risky. The process B_t and S_t satisfy the following stochastic differential equations (SDE):

$$dB_t = r_t B_t dt, \quad B_0 = 0, \tag{2.1.1}$$

$$dS_t = (r_t + \lambda)S_t dt + \sigma_s S_t dW^{\mathbb{P}}_{t,S}, \quad S_0 = 1, \tag{2.1.2}$$

where σ_s denotes the time-independent volatility of the stock, and r_t denotes the stochastic interest rate. The expected return of B_t at time t is r_t. Additionally, the drift term $(r_t + \lambda)$ of S_t implies that the stock yields a constant extra return λ on the top of the expected interest return. The solution of B_t can be derived as follows:

$$B_t = \exp\left\{ \int_0^t r_u du \right\}. \tag{2.1.3}$$

As the dynamic of the stock price process S_t follows a geometric Brownian motion, the solution of S_t is given by

$$S_t = \exp\left\{ \int_0^t r_u du + \left(\lambda - \frac{1}{2}\sigma_s^2\right)t + \sigma_s W^{\mathbb{P}}_{t,S} \right\}. \tag{2.1.4}$$

The Vasicek short rate r_t incorporates mean reversion, and r_t follows the Ornstein-Uhlenbeck process with the following SDE:

$$dr_t = \kappa(\bar{r} - r_t)dt + \sigma_r dW^{\mathbb{P}}_{t,r}. \tag{2.1.5}$$

where $\bar{r}$ is the reversion level of interest rate in the long run, and κ is the speed of mean-reversion. In other words, mean-reversion implies that the short rate r_t will be pulled back to the level $\bar{r}$ at rate κ. Assume that $W^{\mathbb{P}}_{t,S}$ and $W^{\mathbb{P}}_{t,r}$ are two correlated Brownian motions with a correlation parameter $\rho \in (-1, 1)$, and the following properties

$$\mathrm{Cov}(W^{\mathbb{P}}_{t,S}, W^{\mathbb{P}}_{t,r}) = \rho_{r,S} t, \tag{2.1.6}$$

and

$$dW^{\mathbb{P}}_{t,S} dW^{\mathbb{P}}_{t,r} = \rho_{r,S} dt \tag{2.1.7}$$

holds for $t \in [0, T]$. In order to demonstrate the correlation explicitly in the interest rate model (2.1.5), we use the factor model developed by Primbs (2016) to rewrite the Brownian motion $W^{\mathbb{P}}_{t,r}$ as follows:

$$dW^{\mathbb{P}}_{t,r} = \rho_{r,S} dW^{\mathbb{P}}_{t,S} + \sqrt{1 - \rho^2_{r,S}} dW^{\mathbb{P}}_{t,\perp}, \tag{2.1.8}$$

where $W^{\mathbb{P}}_{t,\perp}$ is a new Brownian motion that is independent of $W^{\mathbb{P}}_{t,S}$, i.e. $W^{\mathbb{P}}_{t,\perp} \perp\!\!\!\perp W^{\mathbb{P}}_{t,S}$. The factor model by Primbs (2016) allows us to model a normally distributed variable as a sum of components of multiple random variables. These components should also follow normal distributions, and the squared coefficient of each components should sum up to one. In our case, three Brownian motions in (2.1.8) are normally distributed, and the coefficient $dW^{\mathbb{P}}_{t,S}$ and $dW^{\mathbb{P}}_{t,\perp}$ satisfies

$$\rho^2_{r,S} + \left(\sqrt{1-\rho^2_{r,S}}\right)^2 = 1. \tag{2.1.9}$$

Substitute (2.1.8) into (2.1.5), and we obtain the interest rate process

$$dr_t = \kappa(\bar{r} - r_t)dt + \sigma_r\left(\rho_{r,S}dW^{\mathbb{P}}_{t,S} + \sqrt{1-\rho^2_{r,S}}dW^{\mathbb{P}}_{t,\perp}\right) \tag{2.1.10}$$

under $W^{\mathbb{P}}_{t,S}$ and $W^{\mathbb{P}}_{t,\perp}$ instead. The solution of the SDE dr_t is shown in the following proposition:

Proposition 1 (**Vasicek Short Rate**)**.** *Given the SDE of the Vasicek short rate in* (2.1.10)*, its solution is given by*

$$r_t = e^{-\kappa(t-s)}r_s + \bar{r}\left(1 - e^{-\kappa(t-s)}\right) + \sigma_r e^{-\kappa t}\int_s^t e^{\kappa u}dW^{\mathbb{P}}_{u,r}, \tag{2.1.11}$$

where r_t is normally distributed with mean

$$\mathbb{E}[r_t] = e^{-\kappa(t-s)}r_s + \bar{r}\left(1 - e^{-\kappa(t-s)}\right), \tag{2.1.12}$$

and variance

$$Var[r_t] = \frac{\sigma_r^2}{2\kappa}\left(1 - e^{-2\kappa(t-s)}\right). \tag{2.1.13}$$

Proof. See Mamon (2004) for the derivation of r_t by applying Ito's Lemma. □

Let X_t be the wealth process of a portfolio consisting of stocks S_t and money market accounts B_t. Set up hedging strategies $\phi = \{\phi_{t,S}, \phi_{t,B}\}$. Assume that investors can invest $\phi_{t,S}$ and $\phi_{t,B}$ amount of money in stocks and money market accounts respectively. If the portfolio satisfies the self-financing condition, then the wealth process is as follows:

$$dX_t = \phi_{t,S}dS_t + \phi_{t,B}dB_t. \tag{2.1.14}$$

Equivalently,

$$X_t = X_0 + \int_0^t \phi_{u,S}dS_u + \int_0^t \phi_{u,B}dB_u, \tag{2.1.15}$$

where $X_0 \in \mathbb{R}_+$ is the initial endowment of investors, and the initial value of the portfolio as well. A self-financing portfolio implies that investors can change their hedging strategy from $\phi_{t-\Delta t}$ to ϕ_t without withdrawing or investing additional capital in the portfolio. The portfolio value X_t at time t is determined by the changes in both hedging strategies and the value of assets from the initial time 0 to time t.

2.2 Risk Neutral Pricing

Black and Scholes (1973) approached the option pricing model based on arbitrary free pricing. Asset pricing theory suggests that asset prices can be determined by discounting the expected value of future asset payoffs. To price an asset, we need to compute the conditional expectation of asset payoffs with respect to its filtration under a probability measure. In the real world, the real measure is generally unknown, and the conditional expectation often can not be computed straight forwards under the real-world measure. However, we can construct a new probability measure that is equivalent to the original one, and then compute asset prices where discounted price processes are martingales with conditional expectations equal to one under the new measure. This motivates the following definition of the equivalent martingale measure:

Definition 1 (**Equivalent Martingale Measure**). *A probability measure $\mathbb{Q}$ is said be an equivalent martingale measure of another probability measure $\mathbb{P}$ if the following conditions are satisfied:*

- *$\mathbb{Q}$ is equivalent to $\mathbb{P}$ $(Q \sim P)$.*
- *The discounted price process is a martingale under the measure $\mathbb{Q}$.*

Assume that we are in a risk-neutral world where arbitrage opportunities do not exist. The absence of arbitrage implies that investors cannot profit from price differences of assets in different markets. Under the arbitrage-free condition, a risk-neutral measure must exist, and its existence is guaranteed by the theorem below:

Theorem 2 (**First Fundamental Theorem of Asset Pricing**). *There is no arbitrary opportunity in a market if and only if there exists at least one risk-neutral measure.*

Now let us introduce a new probability measure $\mathbb{Q}$ defined on the same probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]})$ as the measure $\mathbb{P}$. The measure $\mathbb{Q}$ is the risk-neutral measure that is equivalent to the original measure $\mathbb{P}$. To find the discount price process, let us first introduce the notion of normalisation. The financial market can be normalised by dividing all asset processes by a numeraire. As the money market account $B_t > 0$ is strictly positive, we are allowed to choose it as the numeraire. B_t can also be considered as the unit of a price, and we can compute the relative prices of other assets with respect to B_t. The discounted price process of the stock is given by

$$d\frac{S_t}{B_t} = \lambda\frac{S_t}{B_t}dt + \sigma_s\frac{S_t}{B_t}dW_{t,S}^{\mathbb{P}} \tag{2.2.1}$$

under the original measure $\mathbb{P}$.

Now we proceed to find the discounted price process (2.2.1) under the risk-neutral measure $\mathbb{Q}$. Let $\zeta_{t,S}$ be a Radon–Nikodym exponent, see Halmos et al. (1949). By Girsanov's theorem, see Girsanov (1960), the Brownian motion

$$W_{t,S}^{\mathbb{Q}} = W_{t,S}^{\mathbb{P}} + \int_0^t \zeta_{u,S}du \tag{2.2.2}$$

is a standard Brownian motion under $\mathbb{Q}$. Equivalently,

$$dW_{t,S}^{\mathbb{Q}} = dW_{t,S}^{\mathbb{P}} + \zeta_{t,S}dt. \tag{2.2.3}$$

Substitute (2.2.3) into (2.2.1), and we obtain the discounted process of the stock

$$d\frac{S_t}{B_t} = \frac{S_t}{B_t}[(\lambda - \zeta_{t,S}\sigma_s)dt + \sigma_s dW_{t,S}^{\mathbb{Q}}] \tag{2.2.4}$$

under $\mathbb{Q}$. The discounted price process (2.2.4) is a martingale with a zero-drift term. Hence, we can rewrite (2.2.4) as follows:

$$d\frac{S_t}{B_t} = \frac{S_t}{B_t}[\sigma_s dW_{t,S}^{\mathbb{Q}}], \tag{2.2.5}$$

where the drift term vanished. Additionally, we obtain the Radon–Nikodym exponent as follows:

$$\zeta_{t,S} = \frac{\lambda}{\sigma_s}. \tag{2.2.6}$$

Note that $\zeta_{t,S}$ is time-independent. $\zeta_{t,S}$ is also known as the market price of risk or the Sharpe ratio (see Sharpe (1966)), which indicates the extra expected return that investors would demand in order to bear extra risks in the portfolio. If the market price of risk $\zeta_{t,S}$ is large, then investors are willing to invest more capitals in stocks.

2.3 Pricing Kernel

In the risk-neutral world, there exist at least one stochastic discount factors, which can be used to discount expected payoffs of assets. Stochastic discount factors are also known as pricing kernels. Define a pricing kernel M_t satisfying

$$M_t = \exp\left\{ -\int_0^t r_u du \right\} \frac{d\mathbb{Q}}{d\mathbb{P}}, \quad M_0 = 1, \tag{2.3.1}$$

where $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is the Radon–Nikodym derivative, see Halmos et al. (1949). Under the original measure, the discounted expected value of a portfolio X_t is equal to its initial value X_0. This can be shown in the following expression:

$$X_0 = \mathbb{E}^{\mathbb{P}}[M_t X_t]. \tag{2.3.2}$$

By the Girsanov theorem, the Radon–Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$ in the pricing kernel (2.3.1). can transfer the expectation (2.3.2) from the original measure $\mathbb{P}$ to the risk neutral measure $\mathbb{Q}$ as follows:

$$\begin{aligned} X_0 &= \mathbb{E}^{\mathbb{P}}[M_t X_t] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left\{ -\int_0^t r_u du \right\} X_t \right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\frac{X_t}{B_t} \right], \end{aligned} \tag{2.3.3}$$

where the discounted wealth process $\frac{X_t}{B_t}$ is a martingale under the measure $\mathbb{Q}$. The purposes of introducing the pricing kernel M_t are to discount the wealth process, and to change the process from the measure $\mathbb{P}$ to $\mathbb{Q}$.

Now let us examine the Radon–Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$ in (2.3.1). In section 2.1, we introduced two correlated Brownian motions $W = \{W^{\mathbb{P}}_{t,S}, W^{\mathbb{P}}_{t,r}\}$ in the Black-Scholes-Vasicek Model. Furthermore, we showed that $W^{\mathbb{P}}_{t,r}$ can be expressed as $W^{\mathbb{P}}_{t,S}$ and $W^{\mathbb{P}}_{t,\perp}$ in equation (2.1.8). Thus, we can include the Brownian motion set $W = \{W^{\mathbb{P}}_{t,S}, W^{\mathbb{P}}_{t,\perp}\}$ in the wealth process X_t, instead of the original set $W = \{W^{\mathbb{P}}_{t,S}, W^{\mathbb{P}}_{t,r}\}$. The Radon–Nikodym derivative changes the wealth process X_t from the measure $\mathbb{P}$ to $\mathbb{Q}$. According to the Radon–Nikodym theorem, it is given by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathscr{F}_t} = \exp\left\{ -\frac{1}{2}\int_0^t {\zeta_{u,S}}^2 du - \int_0^t \zeta_{u,S} dW^{\mathbb{P}}_{t,S} - \frac{1}{2}\int_0^t {\zeta_{u,\perp}}^2 du - \int_0^t \zeta_{u,\perp} dW^{\mathbb{P}}_{t,\perp} \right\}, \tag{2.3.4}$$

where $\zeta_{t,S}$ and $\zeta_{t,\perp}$ are the Radon-Nikodym exponents for the measure $\mathbb{Q}$ with respect to $\mathbb{P}$. By the Girsanov theorem, the Brownian motions

$$dW_{t,S}^{\mathbb{Q}} = \zeta_{t,S}dt + dW_{t,S}^{\mathbb{P}}, \tag{2.3.5}$$

and

$$dW_{t,\perp}^{\mathbb{Q}} = \zeta_{t,\perp}dt + dW_{t,\perp}^{\mathbb{P}} \tag{2.3.6}$$

are standard Brownian motions under the measure $\mathbb{Q}$.

Insert equation (2.3.4) into (2.3.1), and we can rewrite the pricing kernel as follows:

$$\begin{aligned} &M_t \\ &= \exp\left\{ -\int_0^t r_u du - \frac{1}{2}\int_0^t \zeta_{u,S}^2 du - \int_0^t \zeta_{u,S} dW_{t,S}^{\mathbb{P}} - \frac{1}{2}\int_0^t {\zeta_{u,\perp}}^2 du - \int_0^t \zeta_{u,\perp} dW_{t,\perp}^{\mathbb{P}} \right\}. \end{aligned} \tag{2.3.7}$$

Note that we already computed $\zeta_{t,S}$ in (2.2.6). Thus, we proceed to compute another Radon-Nikodym exponent $\zeta_{t,\perp}$. The approach to determine $\zeta_{t,\perp}$ is using the change of measure for interest rate process r_t. Assume that the interest rate process is

$$\begin{aligned} dr_t &= \kappa(\bar{r}^* - r_t)dt + \sigma_r dW_{t,r}^{\mathbb{Q}} \\ &= \kappa(\bar{r}^* - r_t)dt + \sigma_r\left(\rho_{r,S}dW_{t,S}^{\mathbb{Q}} + \sqrt{1-\rho_{r,S}^2}dW_{t,\perp}^{\mathbb{Q}}\right) \end{aligned} \tag{2.3.8}$$

under the measure $\mathbb{Q}$, where $\bar{r}^*$ is the new short rate parameter. By the change of measure approach, we can rewrite the interest rate process in (2.1.10) from the measure $\mathbb{P}$ to $\mathbb{Q}$ as follows:

$$\begin{aligned} &dr_t \\ &= \kappa(\bar{r} - r_t)dt + \sigma_r\left(\rho_{r,S}dW_{t,S}^{\mathbb{P}} + \sqrt{1-\rho_{r,S}^2}dW_{t,\perp}^{\mathbb{P}}\right) \\ &= \left[\kappa(\bar{r} - r_t) - \sigma_r\left(\rho_{r,S}\zeta_{t,S} + \sqrt{1-\rho_{r,S}^2}\zeta_{t,\perp}\right)\right]dt + \sigma_r\left(\rho_{r,S}dW_{t,S}^{\mathbb{Q}} + \sqrt{1-\rho_{r,S}^2}dW_{t,\perp}^{\mathbb{Q}}\right). \end{aligned} \tag{2.3.9}$$

Set the drift terms in (2.3.8) and (2.3.9) to be equal, and it yields

$$\bar{r}^* = \bar{r} - \frac{\sigma_r\left(\rho_{r,S}\zeta_{t,S} + \sqrt{1-\rho_{r,S}^2}\zeta_{t,\perp}\right)}{\kappa}. \tag{2.3.10}$$

As the interest rate process is a martingale under the measure $\mathbb{Q}$, the drift term in (2.3.9) should vanish. Hence, we obtain the second Radon-Nikodym exponent $\zeta_{t,\perp}$ as follows:

$$\zeta_{t,\perp} = \frac{1}{\sqrt{1-\rho_{r,S}^2}}\bigg(\frac{(\bar{r}^* - \bar{r})\kappa}{\sigma_r} - \frac{\rho_{r,S}\lambda}{\sigma_s}\bigg). \tag{2.3.11}$$

Notice that both $\zeta_{t,\perp}$ (2.3.11) and $\zeta_{t,S}$ (2.2.6) are time-independent. Thus, we change the notations of Radon-Nikodym exponents as ζ_S and $\zeta_\perp$ for the remainder of this paper. Now we can rewrite the pricing kernel in (2.3.7) as follows:

$$M_t = \exp\bigg\{-\int_0^t r_u du - \frac{1}{2}\zeta_S^2 t - \frac{1}{2}\zeta_\perp^2 t - \zeta_S W_{t,S}^{\mathbb{P}} - \zeta_\perp W_{t,\perp}^{\mathbb{P}}\bigg\}. \tag{2.3.12}$$

2.4 Portfolio Wealth and Zero-Coupon Bonds

In section 2.1, we constructed a portfolio consisting of a stock and a money market account. Now let us introduce a new asset zero-coupon bond in the portfolio X_t. Let $P_{(t,T)}$ denote the price of a zero-coupon bond at time $t \in [0,T]$, and the payment is equal to 1 at the time of maturity T, i.e. $P_{(T,T)} = 1$. The price process $P_{(t,T)}$ is based on the underlying interest rate process (2.3.8) in the Vasicek model, and the price process of bonds is given by

$$\frac{P_{(t,T)}}{B_t} = \mathbb{E}^{\mathbb{Q}}\bigg[\frac{P_{(t,T)}}{B_T}\bigg|\mathscr{F}_t\bigg] = \mathbb{E}^{\mathbb{Q}}\bigg[\frac{1}{B_T}\bigg|\mathscr{F}_t\bigg], \tag{2.4.1}$$

where $\frac{P_{(t,T)}}{B_t}$ is a martingale under $\mathbb{Q}$. Rearrange the above equation, and we obtain

$$P_{(t,T)} = \mathbb{E}^{\mathbb{Q}}\bigg[\frac{B_t}{B_T}\bigg|\mathscr{F}_t\bigg] = \mathbb{E}^{\mathbb{Q}}\bigg[\exp\bigg\{-\int_t^T r_u du\bigg\}\bigg|\mathscr{F}_t\bigg]. \tag{2.4.2}$$

Notice that the price process $P_{(t,T)}$ is computed as the conditional expectation with respect to the filtration $\mathscr{F}_t$ under the risk-neutral measure $\mathbb{Q}$. Equation (2.4.2) estimates the future payoffs of the zero-coupon bond, given the information up to the current time t.

To find the bond price $P_{(t,T)}$, we first compute the integral term $R_{(t,T)} = \int_t^T r_u du$ in (2.4.2), and its distribution is shown in the following proposition:

Proposition 3. *If the interest rate r_t follows the Vasicek model in* (2.3.8)*, then the random variable $R_{(t,T)}$ is normally distributed with mean*

$$\mu_{R_{(t,T)}} = \mathbb{E}\left[\int_t^T r_u du \middle| \mathscr{F}_t\right] = \bar{r}^*(T-t) + (r_t - \bar{r}^*)K_{(t,T)}, \tag{2.4.3}$$

and variance

$$\sigma^2_{R_{(t,T)}} = Var\left[\int_t^T r_u du \middle| \mathscr{F}_t\right] = -\frac{\sigma_r^2}{\kappa^2}(K_{(t,T)} - T + t) - \frac{\sigma_r^2}{2\kappa}K^2_{(t,T)}, \tag{2.4.4}$$

where

$$K_{(t,T)} = \frac{1 - e^{-\kappa(T-t)}}{\kappa}. \tag{2.4.5}$$

Proof. See Privault (2013) and Dana & Jeanblanc (2007) for the derivation of the distribution of $\int_t^T r_u du$. □

Now we know the distribution of the integral term $\int_t^T r_u du$ in proposition 3, and then we can obtain the price of zero-coupon bonds. The bond price is shown in the following proposition:

Proposition 4. *Let T be the time of maturity of a zero-coupon bond. If the interest rate r_t follows the Vasicek short rate model, then the price of a zero-coupon bond $P_{(t,T)}$ at time $t \in [0,T]$ has the following form:*

$$P_{(t,T)} = A_{(t,T)} e^{-r_t K_{(t,T)}}, \tag{2.4.6}$$

where

$$A_{(t,T)} = \exp\left\{\left(\bar{r}^* - \frac{\sigma_r^2}{2\kappa^2}\right)(K_{(t,T)} - T + t) - \frac{\sigma_r^2}{4\kappa}K^2_{(t,T)}\right\}. \tag{2.4.7}$$

Additionally, the dynamics of the zero-coupon bond is given by

$$dP_{(t,T)} = r_t P_{(t,T)} dt - \sigma_r K_{(t,T)} P_{(t,T)} dW^{\mathbb{Q}}_{t,r} \tag{2.4.8}$$

under the risk-neutral measure $\mathbb{Q}$.

Proof. See Mamon (2004) and Privault (2013) for the derivation of zero-coupon bonds. □

Given the bond price process $P_{(t,T)}$ in (2.4.8), we divide the process by the numeraire B_t. Then, we obtain

$$\begin{aligned} d\frac{P_{(t,T)}}{B_t} &= -\sigma_r K_{(t,T)} \frac{P_{(t,T)}}{B_t} dW^{\mathbb{Q}}_{t,r} \\ &= -\sigma_r K_{(t,T)} \left(\rho_{r,S} \frac{P_{(t,T)}}{B_t} dW^{\mathbb{Q}}_{t,S} + \sqrt{1-\rho^2_{r,S}} \frac{P_{(t,T)}}{B_t} dW^{\mathbb{Q}}_{t,\perp} \right), \end{aligned} \tag{2.4.9}$$

where $\frac{P_{(t,T)}}{B_t}$ is a martingale with a zero-drift term under the risk-neutral measure $\mathbb{Q}$.

After including a zero-coupon bond in the portfolio, we need to add the bond price process to the previous wealth process in (2.1.14) and (2.1.15). Let $\phi = \{\phi_{t,S}, \phi_{t,B}, \phi_{t,P}\}$ be the new set of hedging strategies. Then, we obtain the new wealth process X_t as follows:

$$dX_t = \phi_{t,S} dS_t + \phi_{t,B} dB_t + \phi_{t,P} dP_{(t,T)}. \tag{2.4.10}$$

Equivalently,

$$X_t = X_0 + \int_0^t \phi_{u,S} dS_u + \int_0^t \phi_{u,B} dB_u + \int_0^t \phi_{u,P} dP_{(u,T)}. \tag{2.4.11}$$

If we divide the wealth process X_t by the numeraire B_t, then it yields

$$\frac{X_t}{B_t} = \frac{X_0}{B_0} + \int_0^t \phi_{u,S} d\frac{S_u}{B_u} + \int_0^t \phi_{u,P} d\frac{P_{(u,T)}}{B_u}, \tag{2.4.12}$$

where $\frac{X_t}{B_t}$ is a martingale under the measure $\mathbb{Q}$. According to the Martingale representation theorem, see Jacka (1992), the martingale $\frac{X_t}{B_t}$ has a unique representation in terms of stochastic integral. As we need to find a trading strategy $\phi = \{\phi_{t,S}, \phi_{t,B}, \phi_{t,P}\}$ that replicates the payoff, the Martingale representation theorem ensures that the trading strategy exists, and the strategy is unique.

Chapter 3

Utility Maximisation and the Martingale Method

In this chapter, we formulate the dynamic optimisation problem where investors aim to maximise their expected utilities given a benchmark. Then, we explain the Martingale method to find a general solution of a utility function. Last, we present an explicit example of applying the Martingale method to compute the optimal wealth under a power utility.

3.1 Optimisation Problem

The optimal investment problem was first approached by Merton (1969) who solved the problem using the dynamic programming approach. In this paper, we consider an alternative approach called the Martingale method which was developed by Cox & Huang (1989) and Karatzas et al. (1987).

Let $U(x)$ denote the utility function. Consider an agent who invests an initial capital $X_0 \in \mathbb{R}_+$ in a portfolio, and he aims to maximise his expected utility by choosing dynamic hedging strategies at the terminal time T. In this paper, we extend the general utility maximisation problem introduced by Merton (1969) with a benchmark $Y_T \in \mathbb{R}_+$. The purpose of setting up a benchmark is to measure the performance of the portfolio. The benchmark indicates the minimum value of the portfolio wealth that the agent intends to achieve. The utility function $U(X_T, Y_T)$ has two variables including the portfolio wealth X_T and the benchmark Y_T. The dynamic optimisation problem can be stated as follows:

$$\max_{X_T} \quad \mathbb{E}[U(X_T, Y_T)]. \tag{3.1.1}$$

Note that the utility of the agent should be diminishing. Thus, we assume that the utility function should satisfy the conditions stated in the following assumption:

Assumption 5. *The utility function $U(X_T, Y_T)$ satisfies Inada conditions with respect to X_T for any value of Y_T :*

- $U'_{X_T}(0, Y_T) \triangleq \lim_{X_T \downarrow 0} U'_{X_T}(X_T, Y_T) = \infty$
- $U'_{X_T}(\infty, Y_T) \triangleq \lim_{X_T \to \infty} U'_{X_T}(X_T, Y_T) = 0$

The above assumption ensures that the utility function $U(X_T, Y_T)$ is continuous, monotonically increasing, and strictly concave.

3.2 The Martingale Method

Let us now process to use the martingale method to solve the optimisation problem in (3.1.1). In section 2.2, we explained the notion of risk-neutral pricing. Let us first consider the complete market case, and then extend the method to the incomplete market. This motivates the following theorem about the market completeness:

Theorem 6 (Second Fundamental Theorem of Asset Pricing). *The market is complete if and only if there exists a unique equivalent martingale measure. Otherwise, the market is said to be incomplete.*

In the complete market, there is a unique risk-neutral measure $\mathbb{Q}$ such that the discounted wealth process is a martingale. Because of the arbitrage-free condition in the complete market, the law of one price is satisfied. In other words, all assets with the same payoffs have a unique price. These assets can perfectly replicate the payoffs of the portfolio. Under the measure $\mathbb{Q}$, the portfolio wealth X_T has a unique price. We can set up a budge constrain such that the expectation of discounted portfolio wealth at time T is equal to the initial value of the portfolio X_0. Hence, we transfer the dynamic optimisation problem in (3.1.1) into a static optimisation problem as follows:

$$\max_{X_T} \quad \mathbb{E}[U(X_T, Y_T)], \tag{3.2.1a}$$

$$s.t. \quad \mathbb{E}[M_T X_T] = X_0, \tag{3.2.1b}$$

where $(3.2.1b)$ is the budge constrain for the agent.

To solve the above optimisation problem, we use the Lagrangian method such that the optimisation with a constrain in (3.2.1) can be transformed into an unconstrained optimisation. Set up the Lagrangian function as follows:

$$\mathscr{L}(X_T, \eta) = \mathbb{E}[U(X_T, Y_T)] + \eta(X_0 - \mathbb{E}[M_T X_T]), \tag{3.2.2}$$

where η is the Lagrangian multiplier, and $X_0 = 1$. Assume that Lagrange $\mathscr{L}(X_T, \eta)$ is Fréchet-differentiable in infinite-dimensional spaces, see Mathias (1992) or Al-Mohy & Higham (2009) for the definition of Fréchet derivative. To understand the intuition of taking the derivative of X_T, let us assume that we approximate the continuous variable X_T with a discrete variable in a finite space $\mathbb{R}_n$. Expectation term $\mathbb{E}[U(X_T, Y_T)]$ in (3.2.2) can be interpreted as the summation of the probabilities of X_T in each state of the world times the values of utilities $U(X_T, Y_T)$. Note that $U(X_T, Y_T)$ is a function in $\mathbb{R}_n$ for the discrete approximation. In this case, the notion of taking the derivative of X_T can be seen as a normal vector differentiation in $\mathbb{R}_n$.

Take the first-order condition of the Lagrangian function $\mathscr{L}(X_T, \eta)$, and then set the derivative to zero. It yields:

$$U'_{X_T}(X_T, Y_T) = \eta M_T. \tag{3.2.3}$$

Note that (3.2.3) is an identity rather than an equality. The inverse of the utility in (3.2.3) equals to ηM_T in each state of the world. Let $I(., Y_T)$ denote an inverse function with respect to X_T. Assume that the utility function in (3.2.3) satisfies Inada conditions, and $U(X_T, Y_T)$ with respect to X_T is a monotonically increasing and concave function. Hence, we can solve (3.2.3) with respect to X_T, and then obtain the optimal wealth at time T as follows:

$$\hat{X}_T = I(\eta M_T, Y_T), \tag{3.2.4}$$

Note that the Lagrangian multiplier η is undetermined in this step. However, we can compute η by inserting the optimal wealth (3.2.4) into the budget constrain (3.2.1b). The Lagrangian multiplier η is the solution of

$$\mathbb{E}[M_T I(\eta M_T, Y_T)] = X_0. \tag{3.2.5}$$

After obtaining the optimal terminal wealth in (3.2.4), we can also compute the optimal wealth at any arbitrage time $t \in [0, T]$ from the optimal wealth at time T. Because of the law of one price, the conditional expectation of the discounted wealth process at time T given the filtration $\mathscr{F}_t$ should be equal to the discounted wealth at time t. That is,

$$M_t \hat{X}_t = \mathbb{E}\left[M_T \hat{X}_T \middle| \mathscr{F}_t \right], \tag{3.2.6}$$

where M_t is $\mathscr{F}_t$-measurable. Equivalently, we can rewrite the optimal wealth (3.2.6) at time t as follows:

$$\hat{X}_t = \mathbb{E}\left[\frac{M_T}{M_t}\hat{X}_T\middle|\mathscr{F}_t\right]. \tag{3.2.7}$$

Note that the optimal wealth in (3.2.6) and (3.2.7) is under the original measure $\mathbb{P}$. According to the Girsanov theorem, the pricing kernel M_T can transfer the expectation in (3.2.6) and (3.2.7) from the original measure $\mathbb{P}$ to the risk-neutral measure $\mathbb{Q}$, and we obtain:

$$\frac{\hat{X}_t}{B_t} = \mathbb{E}^{\mathbb{Q}}\left[\frac{\hat{X}_T}{B_T}\middle|\mathscr{F}_t\right], \tag{3.2.8}$$

where the discounted optimal wealth process $\frac{\hat{X}_t}{B_t}$ is a martingale under $\mathbb{Q}$.

3.3 Power Utility

In section 3.2, we explained the martingale method to solve the utility optimisation problem. Now let us consider a power utility, and derive the optimal wealth at the terminal time T and an arbitrage time t. The power utility is given by

$$U(\frac{X_T}{Y_T}) = \frac{(\frac{X_T}{Y_T})^{1-\gamma} - 1}{1-\gamma}, \quad \gamma \neq 1, \tag{3.3.1}$$

where the parameter $\gamma > 0$ measures the degree of risk-aversion for agents. The utility function (3.3.1) converges to a logarithmic utility function when $\gamma \to 1$, i.e. γ converges to one. The value of the utility function depends on the ratio between the portfolio wealth and the benchmark. To be more specific, the utility of the agent is higher when the portfolio outperforms the benchmark.

To verify whether the power utility function satisfies the Inada conditions defined in Assumption 5, we compute its first and second-order derivative as follows:

$$U'_{X_T}(\frac{X_T}{Y_T}) = (\frac{X_T}{Y_T})^{-\gamma}\frac{1}{Y_T} > 0, \tag{3.3.2}$$

and

$$U''_{X_T}(\frac{X_T}{Y_T}) = -\gamma(\frac{X_T}{Y_T})^{-\gamma-1}\frac{1}{Y_T^2} < 0. \tag{3.3.3}$$

Equation (3.3.2) and (3.3.3) imply that the power utility is strictly increasing and strictly concave. Thus, the Inada conditions are satisfied.

Now let us proceed to derive the optimal wealth under the power utility. Based on the general solution of the optimal wealth in (3.2.4), we compute the inverse function of the first derivative of the power utility (3.3.2) as follows:

$$I(U'_{X_T}(\frac{X_T}{Y_T}), Y_T) = Y_T(X_T Y_T)^{-\frac{1}{\gamma}}. \tag{3.3.4}$$

Insert the inverse function (3.3.4) into the optimal terminal wealth function (3.2.4), and it yields:

$$\hat{X}_T = I(\eta M_T, Y_T) = (\eta M_T)^{-\frac{1}{\gamma}} Y_T^{1-\frac{1}{\gamma}}, \tag{3.3.5}$$

where the Lagrangian multiplier η is the solution of the budget constrain

$$\mathbb{E}[I(\hat{\eta} M_T, Y_T) M_T] = \eta^{-\frac{1}{\gamma}} \mathbb{E}\left[(M_T Y_T)^{1-\frac{1}{\gamma}}\right] = X_0. \tag{3.3.6}$$

Rearrange equation (3.3.6), and we get the Lagrangian multiplier as follows:

$$\eta = \left(\frac{X_0}{\mathbb{E}[(M_T Y_T)^{1-\frac{1}{\gamma}}]}\right)^{-\gamma}. \tag{3.3.7}$$

Then, insert the Lagrangian multiplier η that we derived in (3.3.7) into the optimal wealth (3.3.5) . This yields:

$$\begin{aligned} \hat{X}_T &= (\hat{\eta} M_T)^{-\frac{1}{\gamma}} Y_T^{1-\frac{1}{\gamma}} \\ &= \frac{X_0}{\mathbb{E}[(M_T Y_T)^{1-\frac{1}{\gamma}}]} M_T^{-\frac{1}{\gamma}} Y_T^{1-\frac{1}{\gamma}}. \end{aligned} \tag{3.3.8}$$

Hence, we derived the optimal terminal wealth $\hat{X}_T$ in (3.3.8) under the power utility by the martingale method. Notice that the optimal terminal wealth (3.3.8) is determined by the expectation of the discounted benchmark, the pricing kernel M_T, and the benchmark itself. For the remainder of this chapter, we will discuss three concrete examples about the optimal wealth under different benchmarks including a cash, a stock, and an inflation-indexed bond benchmark.

Chapter 4

Optimal Portfolio Wealth

In this chapter, we analyse optimal portfolio wealth under the power utility using the Martingale method. We present three cases by choosing different benchmarks, including a constant, a stock, and an inflation-indexed bond benchmark. We first discuss optimal wealth at a terminal time, and then extend the problem to find optimal wealth at an arbitrage time before the final time.

4.1 Optimal Wealth under a Cash Benchmark

Consider an investor who aims to compare the portfolio wealth with a constant number. Let $Y_C \in \mathbb{R}_+$ denote a constant benchmark. Assume that the initial capital of the portfolio is 1, i.e. $X_0 = 1$. To obtain the optimal terminal wealth $\hat{X}_{T,C}$ under the constant benchmark, we substitute the benchmark term Y_T into (3.3.8) with Y_C. This yields:

$$\begin{aligned}\hat{X}_{T,C} &= \frac{X_0}{\mathbb{E}[(M_T Y_C)^{1-\frac{1}{\gamma}}]} M_T^{-\frac{1}{\gamma}} Y_C^{1-\frac{1}{\gamma}} \\ &= \frac{1}{Y_C^{1-\frac{1}{\gamma}} \mathbb{E}[M_T^{1-\frac{1}{\gamma}}]} M_T^{-\frac{1}{\gamma}} Y_C^{1-\frac{1}{\gamma}} \\ &= \frac{M_T^{-\frac{1}{\gamma}}}{\mathbb{E}[M_T^{1-\frac{1}{\gamma}}]}.\end{aligned} \tag{4.1.1}$$

Notice that the term Y_C vanished in the optimal wealth (4.1.1), which implies that the constant benchmark does not affect the optimal terminal wealth $\hat{X}_{T,C}$. The pricing kernel M_t and its expectation determine the value of the optimal wealth. In section 4.3.14, we derived the pricing kernel (2.3.12) to discount the portfolio wealth in (2.4.10). In this section, the portfolio wealth $\hat{X}_{T,C}$ and the pricing kernel M_t are still the same as we defined in (2.4.10) and (2.3.12).

To derive the optimal terminal wealth $\hat{X}_{T,C}$, we compute the pricing kernel M_t and its expectation respectively. Now let us consider the expectation of the pricing kernel first. We have

$$\begin{aligned}
&\mathbb{E}[M_T^{1-\frac{1}{\gamma}}] \\
&= \mathbb{E}\bigg[\exp\bigg\{-\int_0^T r_u du - \frac{1}{2}\zeta_S^2 T - \frac{1}{2}\zeta_\perp^2 T - \zeta_S W_{t,S}^{\mathbb{P}} - \zeta_\perp W_{t,\perp}^{\mathbb{P}}\bigg\}^{1-\frac{1}{\gamma}}\bigg] \\
&= \exp\bigg\{-\frac{1}{2}\zeta_S^2 T - \frac{1}{2}\zeta_\perp^2 T\bigg\}^{1-\frac{1}{\gamma}} \mathbb{E}\bigg[\exp\bigg\{-\int_0^T r_u du - \zeta_S W_{t,S}^{\mathbb{P}} - \zeta_\perp W_{t,\perp}^{\mathbb{P}}\bigg\}^{1-\frac{1}{\gamma}}\bigg].
\end{aligned} \tag{4.1.2}$$

Observe that equation (4.1.2) consists of a deterministic term and an expectation term of an exponential variable. To compute the expectation term in (4.1.2), we define a random variable J such that

$$J = -\int_0^T r_u du - \zeta_S W_{T,S}^{\mathbb{P}} - \zeta_\perp W_{T,\perp}^{\mathbb{P}}. \tag{4.1.3}$$

The variable J is normally distributed with mean μ_J and variance σ_J^2. That is,

$$J \sim \mathcal{N}(\mu_J, \sigma_J^2). \tag{4.1.4}$$

Then the expectation is the first order of its moment generating function. Thus,

$$\mathbb{E}[e^J] = \exp\bigg\{\mu_J + \frac{1}{2}\sigma_J^2\bigg\}. \tag{4.1.5}$$

Now let us derive the mean μ_J and the variance σ_J^2 respectively. Note that the Brownian motion $W_{T,S}^{\mathbb{P}}$ and $W_{T,\perp}^{\mathbb{P}}$ are normally distributed with mean 0 and variance 1. The expectation of J is computed as follows:

$$\begin{aligned}
\mu_J &= \mathbb{E}\bigg[-\int_0^T r_u du - \zeta_S W_{T,S}^{\mathbb{P}} - \zeta_\perp W_{T,\perp}^{\mathbb{P}}\bigg] \\
&= \mathbb{E}\bigg[-\int_0^T r_u du\bigg] \\
&= -\mu_{R_{(0,T)}},
\end{aligned} \tag{4.1.6}$$

where the distribution of the integral term $\int_0^T r_u du$ is shown in Proposition 3, and the expectation $\mu_{R_{(0,T)}}$ can be found in (2.4.3). Now we proceed to compute the variance of J. We have

$$\begin{aligned}\sigma_J^2 &= \mathrm{Var}\bigg[-\int_0^T r_u du - \zeta_S W^{\mathbb{P}}_{T,S} - \zeta_\perp W^{\mathbb{P}}_{T,\perp}\bigg]\\ &= \mathrm{Var}\bigg[\int_0^T r_u du\bigg] + \mathrm{Var}[\zeta_S W^{\mathbb{P}}_{T,S}] + \mathrm{Var}[\zeta_\perp W^{\mathbb{P}}_{T,\perp}] + 2\mathrm{Cov}\bigg[\int_0^T r_u du, \zeta_S W^{\mathbb{P}}_{T,S}\bigg]\\ &\quad + 2\mathrm{Cov}\bigg[\int_0^T r_u du, \zeta_\perp W^{\mathbb{P}}_{T,\perp}\bigg] + 2\mathrm{Cov}\bigg[\zeta_S W^{\mathbb{P}}_{T,S}, \zeta_\perp W^{\mathbb{P}}_{T,\perp}\bigg],\end{aligned} \tag{4.1.7}$$

where σ_J^2 is the sum of variances and covariances between each term in J (4.1.3). The sum of the variance terms in (4.1.7) is computed as follows:

$$\mathrm{Var}\bigg[\int_0^T r_u du\bigg] + \mathrm{Var}[\zeta_S W^{\mathbb{P}}_{T,S}] + \mathrm{Var}[\zeta_\perp W^{\mathbb{P}}_{T,\perp}] = \sigma_r^2 \int_0^T K^2_{(u,T)} du + \zeta_S^2 T + \zeta_\perp^2 T, \tag{4.1.8}$$

where the variance of the integral $\int_0^T r_u du$ is shown in (2.4.4). Now we compute the sum of covariance terms in (4.1.7) as follows:

$$\mathrm{Cov}\bigg[\int_t^T r_u du, \zeta_S W^{\mathbb{P}}_{T,S}\bigg] = \mathbb{E}\bigg[\zeta_S^2 \sigma_r \int_0^T K_{(u,T)} dW^{\mathbb{P}}_{u,S} W^{\mathbb{P}}_{u,r}\bigg] = \zeta_S^2 \rho_{r,S} \sigma_r \int_0^T K_{(u,T)} du, \tag{4.1.9}$$

$$\mathrm{Cov}\bigg[\int_t^T r_u du, \zeta_\perp W^{\mathbb{P}}_{T,\perp}\bigg] = \zeta_\perp^2 \sqrt{1-\rho_{r,S}^2}\,\sigma_r \int_0^T K_{(u,T)} du, \tag{4.1.10}$$

and,

$$\mathrm{Cov}\bigg[\zeta_S W^{\mathbb{P}}_{T,S}, \zeta_\perp W^{\mathbb{P}}_{T,\perp}\bigg] = 0. \tag{4.1.11}$$

The Brownian motion $W^{\mathbb{P}}_{T,S}$ and $W^{\mathbb{P}}_{T,\perp}$ are independent. Hence, the covariance between these two Brownian motions are 0 in (4.1.11). Inserting the values of the Radon–Nikodym exponent ζ_S (2.2.6) and $\zeta_\perp$ (2.3.11) into the covariance (4.1.9) and (4.1.10) respectively, we obtain the sum of these two variances as follow:

$$\mathrm{Cov}\bigg[\int_t^T r_u du, \zeta_S W^{\mathbb{P}}_{T,S}\bigg] + \mathrm{Cov}\bigg[\int_t^T r_u du, \zeta_\perp W^{\mathbb{P}}_{T,\perp}\bigg] = \kappa(\bar{r} - \bar{r}^*)\sigma_r \int_0^T K_{(u,T)} du. \tag{4.1.12}$$

Now Insert all above variance terms and covariance terms into the variance σ_J^2 in (4.1.7). This yields:

$$\sigma_J^2 = \sigma_r^2 \int_0^T K_{(u,T)}^2 du + (\zeta_S^2 + \zeta_\perp^2)T + 2\kappa(\bar{r} - \bar{r}^*)\sigma_r \int_0^T K_{(u,T)} du. \qquad (4.1.13)$$

Hence, we found the mean (4.1.6) and the variance (4.1.13) of the variable J. Now we can rewrite the expectation of the pricing kernel (4.1.2) as follows:

$$\mathbb{E}[M_T^{1-\frac{1}{\gamma}}] = \exp\left\{-\frac{1}{2}\zeta_S^2 T - \frac{1}{2}\zeta_\perp^2 T\right\}^{1-\frac{1}{\gamma}} \exp\left\{\left(1-\frac{1}{\gamma}\right)\mu_J + \frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2 \sigma_J^2\right\}. \qquad (4.1.14)$$

Note that the optimal terminal wealth $X_{T,C}$ depends on the pricing kernel M_T and the expectation of the pricing kernel. Therefore, we insert the pricing kernel (2.3.12) and the expectation (4.1.14) into the optimal wealth in (4.1.1). This yields:

$$\begin{aligned}\hat{X}_{T,C} = \exp\bigg\{&\frac{1}{2}\zeta_S^2 T + \frac{1}{2}\zeta_\perp^2 T + \left(\frac{1}{\gamma} - 1\right)\mu_J \\ &- \frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2 \sigma_J^2\bigg\} \exp\left\{\frac{1}{\gamma}\left(\int_t^T r_u du + \zeta_S W_{T,S}^{\mathbb{P}} + \zeta_\perp W_{T,\perp}^{\mathbb{P}}\right)\right\}.\end{aligned} \qquad (4.1.15)$$

Note that the optimal terminal wealth in (4.1.17) is under the original measure $\mathbb{P}$. By the change of measure method, we can transfer the optimal wealth from the measure $\mathbb{P}$ to the risk-neutral measure $\mathbb{Q}$. Thus, we change the Brownian motion $W_{T,S}^{\mathbb{P}}$ and $W_{T,\perp}^{\mathbb{P}}$ from the measure $\mathbb{P}$ to $\mathbb{Q}$ according to the Girsanov theorem, and we obtain:

$$\begin{aligned}\hat{X}_{T,C} = \exp\bigg\{&\left(\frac{1}{2} - \frac{1}{\gamma}\right)\left(\zeta_S^2 T + \zeta_\perp^2 T\right) + \left(\frac{1}{\gamma} - 1\right)\mu_J \\ &- \frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2 \sigma_J^2\bigg\} \exp\left\{\frac{1}{\gamma}\left(\int_0^T r_u du + \zeta_S W_{T,S}^{\mathbb{Q}} + \zeta_\perp W_{T,\perp}^{\mathbb{Q}}\right)\right\}.\end{aligned} \qquad (4.1.16)$$

To simplify calculations, let L_T denote the deterministic term in (4.1.16). Then, we can rewrite the optimal terminal wealth as follows:

$$\hat{X}_{T,C} = L_T \exp\left\{\frac{1}{\gamma}\left(\int_t^T r_u du + \zeta_S W_{T,S}^{\mathbb{Q}} + \zeta_\perp W_{T,\perp}^{\mathbb{Q}}\right)\right\}. \qquad (4.1.17)$$

Hence, we found the optimal terminal wealth $\hat{X}_{T,C}$ under the power utility by applying the martingale approach explained in section 3.2 and 3.3. However, investors also want to maximise the portfolio wealth before the terminal time T. Thus, we proceed to compute the optimal wealth at an arbitrage time $t \in [0, T]$.

In complete markets, the equivalent martingale measure is unique, and the law of one price is satisfied. In section 3.2, equation (3.2.8) implies that the conditional expectation of the discounted wealth process $\frac{\hat{X}_{T,C}}{B_T}$ at the terminal time T given the filtration $\mathscr{F}_t$ equals to the discounted wealth $\frac{\hat{X}_{t,C}}{B_t}$at any arbitrage time $t \in [0, T]$. That is,

$$\frac{\hat{X}_{t,C}}{B_t} = \mathbb{E}^{\mathbb{Q}}\left[\frac{\hat{X}_{T,C}}{B_T}\middle|\mathscr{F}_t\right], \tag{4.1.18}$$

where $\frac{\hat{X}_{t,C}}{B_t}$ and $\frac{\hat{X}_{T,C}}{B_T}$ are martingales under $\mathbb{Q}$. To derive the discounted wealth process at time t, we first compute the discounted terminal wealth based on the optimal wealth in (4.1.17) as follows:

$$\frac{\hat{X}_{T,C}}{B_T} = L_T \text{exp}\left\{\left(\frac{1}{\gamma} - 1\right)\int_0^T r_u du + \frac{1}{\gamma}\left(\zeta_S W_{T,S}^{\mathbb{Q}} + \zeta_\perp W_{T,\perp}^{\mathbb{Q}}\right)\right\}. \tag{4.1.19}$$

Insert (4.1.19) into (4.1.18), and we obtain the optimal wealth at time t as follows:

$$\begin{aligned}\frac{\hat{X}_t}{B_t} = L_T \text{exp}\left\{\left(\frac{1}{\gamma} - 1\right)\int_0^t r_u du + \frac{1}{\gamma}\zeta_S W_{t,S}^{\mathbb{Q}} + \frac{1}{\gamma}\zeta_\perp W_{t,\perp}^{\mathbb{Q}}\right\}\mathbb{E}^{\mathbb{Q}}\Bigg[\text{exp}\left\{\left(\frac{1}{\gamma} - 1\right)\int_t^T r_u du\right.\\ \left. + \frac{1}{\gamma}\zeta_S(W_{T,S}^{\mathbb{Q}} - W_{t,S}^{\mathbb{Q}}) + \frac{1}{\gamma}\zeta_\perp(W_{T,\perp}^{\mathbb{Q}} - W_{t,\perp}^{\mathbb{Q}})\right\}\Bigg|\mathscr{F}_t\Bigg],\end{aligned} \tag{4.1.20}$$

where the Brownian motion $W_{t,S}^{\mathbb{Q}}$ and $W_{t,\perp}^{\mathbb{Q}}$ are $\mathscr{F}_t$-measurable. Hence, we take these two Brownian motions out of the conditional expectation in (4.1.20). Note that the Brownian motion $W_{T,S}^{\mathbb{Q}} - W_{t,S}^{\mathbb{Q}}$ and $W_{T,\perp}^{\mathbb{Q}} - W_{t,\perp}^{\mathbb{Q}}$ are normally distributed with mean 0, and variance $T - t$. That is,

$$(W_{T,S}^{\mathbb{Q}} - W_{t,S}^{\mathbb{Q}})/(W_{T,\perp}^{\mathbb{Q}} - W_{t,\perp}^{\mathbb{Q}}) \sim \mathcal{N}(0, T - t). \tag{4.1.21}$$

To compute the conditional expectation in (4.1.20), let H be a random variable such that

$$H = \left(\frac{1}{\gamma} - 1\right)\int_t^T r_u du + \frac{1}{\gamma}\zeta_S(W_{T,S}^{\mathbb{Q}} - W_{t,S}^{\mathbb{Q}}) + \frac{1}{\gamma}\zeta_\perp(W_{T,\perp}^{\mathbb{Q}} - W_{t,\perp}^{\mathbb{Q}}). \tag{4.1.22}$$

Here we use the same method as in (4.1.5) where we consider the conditional expectation in (4.1.20) as the first order of moment generating function of H. That is,

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}}\left[\exp\left\{\left(\frac{1}{\gamma}-1\right)\int_t^T r_u du+\frac{1}{\gamma}\zeta_S(W^{\mathbb{Q}}_{T,S}-W^{\mathbb{Q}}_{t,S})+\frac{1}{\gamma}\zeta_\perp(W^{\mathbb{Q}}_{T,\perp}-W^{\mathbb{Q}}_{t,\perp})\right\}\middle|\mathscr{F}_t\right]\\&=\exp\left\{\mu_H+\frac{1}{2}\sigma_H^2\right\},\end{aligned}\tag{4.1.23}$$

where the expectation of H is

$$\mu_H=\left(\frac{1}{\gamma}-1\right)\mu_{R_{(0,T)}},\tag{4.1.24}$$

and the variance of H is

$$\sigma_H^2=\left(\frac{1}{\gamma}-1\right)^2\sigma_r^2\int_0^T K^2_{(u,T)}du+\frac{1}{\gamma^2}(\zeta_S^2+\zeta_\perp^2)(T-t)+2\left(\frac{1}{\gamma}-1\right)\frac{1}{\gamma}\kappa(\bar{r}-\bar{r}^*)\sigma_r\int_0^T K_{(u,T)}du.\tag{4.1.25}$$

Here the calculations of μ_H and σ_H^2 are similar to the computation of the expectation and the variance of J in (4.1.6) and (4.1.7). Thus, we present the result above, and the calculation by steps are not shown. After deriving the conditional expectation (4.1.23), we can rewrite the optimal wealth (4.1.20) under the cash benchmark at time t as follows:

$$\begin{aligned}\frac{\hat{X}_{t,C}}{B_t}&=L_T\exp\left\{\mu_H+\frac{1}{2}\sigma_H^2\right\}\exp\left\{\left(\frac{1}{\gamma}-1\right)\int_0^t r_u du+\frac{1}{\gamma}\zeta_S W^{\mathbb{Q}}_{t,S}+\frac{1}{\gamma}\zeta_\perp W^{\mathbb{Q}}_{t,\perp}\right\}\\&=\exp\left\{\left(\frac{1}{2}-\frac{1}{\gamma}\right)\left(\zeta_S^2T+\zeta_\perp^2T\right)+\left(\frac{1}{\gamma}-1\right)\mu_J-\frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2\sigma_J^2\right.\\&\qquad\left.+\mu_H+\frac{1}{2}\sigma_H^2\right\}\exp\left\{\left(\frac{1}{\gamma}-1\right)\int_0^t r_u du+\frac{1}{\gamma}\zeta_S W^{\mathbb{Q}}_{t,S}+\frac{1}{\gamma}\zeta_\perp W^{\mathbb{Q}}_{t,\perp}\right\},\end{aligned}\tag{4.1.26}$$

where the coefficient terms of Brownian motion $W^{\mathbb{Q}}_{t,S}$ and $W^{\mathbb{Q}}_{t,\perp}$ indicate the amount of risk that investors want to take for stocks and zero-coupon bonds. Equation (4.1.26) implies that investors will not invest in stocks if ζ_S is zero. Similarly, investing in zero-coupon bonds is not an optimal strategy when $\zeta_\perp$ is zero.

4.2 Optimal Wealth under a Stock Benchmark

In section 4.1, we discussed the optimal wealth under a constant benchmark. In this section, we choose the stock S_t as a benchmark, and then find the optimal wealth $\hat{X}_{T,S}$ under the stock benchmark. Let $Y_{T,S}$ be a benchmark following the stock process S_t defined in (2.1.4). That is,

$$Y_{T,S} = S_0 \exp\left\{ \int_0^T r_u du + \left(\lambda - \frac{1}{2}\sigma_s^2\right)T + \sigma_s W^{\mathbb{P}}_{T,S} \right\}. \tag{4.2.1}$$

Based on the general solution of the optimal terminal wealth derived in (3.3.8) under the power utility, we have

$$\hat{X}_{T,S} = \frac{(M_T Y_{T,S})^{-\frac{1}{\gamma}} Y_{T,S}}{\mathbb{E}[(M_T Y_{T,S})^{1-\frac{1}{\gamma}}]} = \frac{M_T^{-\frac{1}{\gamma}} Y_{T,S}^{1-\frac{1}{\gamma}}}{\mathbb{E}[(M_T Y_{T,S})^{1-\frac{1}{\gamma}}]}, \tag{4.2.2}$$

where $\hat{X}_{T,S}$ is the optimal terminal wealth under the stock benchmark $Y_{T,S}$. Equation (4.2.2) implies that the value of the optimal terminal wealth $\hat{X}_{T,S}$ depends on three factors, including the benchmark $Y_{T,S}$, the discounted benchmark process $M_T Y_{T,S}$, and the expectation of the discounted benchmark.

To obtain the optimal terminal wealth, let us first compute the expectation of the discounted benchmark. We have:

$$\begin{aligned}&\mathbb{E}[(M_T Y_{T,S})^{1-\frac{1}{\gamma}}]\\ &= \exp\left\{ -\frac{1}{2}\zeta_S^2 T - \frac{1}{2}\zeta_\perp^2 T + \lambda T - \frac{1}{2}\sigma_s^2 T \right\}^{1-\frac{1}{\gamma}} \mathbb{E}\left[\exp\left\{ (-\zeta_S + \sigma_s) W^{\mathbb{P}}_{T,S} - \zeta_\perp W^{\mathbb{P}}_{T,\perp} \right\}^{1-\frac{1}{\gamma}} \right],\end{aligned} \tag{4.2.3}$$

where the pricing kernel is shown in (2.3.12). Note that the Brownian motion $W^{\mathbb{P}}_{T,S}$ and $W^{\mathbb{P}}_{T,\perp}$ is not correlated. That is,

$$\mathrm{Cov}\left((-\zeta_S + \sigma_s) W^{\mathbb{P}}_{T,S}, \zeta_\perp W^{\mathbb{P}}_{T,\perp} \right) = 0. \tag{4.2.4}$$

Similar to the way that we derived the expectation of the pricing kernel consisting of two Brownian motions in (4.1.2), we consider the term

$$(-\zeta_S + \sigma_s) W^{\mathbb{P}}_{T,S} - \zeta_\perp W^{\mathbb{P}}_{T,\perp} \tag{4.2.5}$$

in (4.2.3) as a random variable. Then, the expectation term in (4.2.3) is the first order of the moment generating function of this random variable. This yields:

$$\mathbb{E}\left[\exp\left\{(-\zeta_S+\sigma_s)W^{\mathbb{P}}_{T,S}-\zeta_\perp W^{\mathbb{P}}_{T,\perp}\right\}^{1-\frac{1}{\gamma}}\right]=\exp\left\{\frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2[(-\zeta_S+\sigma_s)^2+\zeta_\perp^2]T\right\}. \tag{4.2.6}$$

Insert the expectation term (4.2.6) into the expectation of the discounted benchmark $M_T Y_{T,S}$ in (4.2.3), and we obtain:

$$\begin{aligned}&\mathbb{E}[(M_T Y_{T,S})^{1-\frac{1}{\gamma}}]\\ &=\exp\left\{-\frac{1}{2}\zeta_S^2 T-\frac{1}{2}\zeta_\perp^2 T+\lambda T-\frac{1}{2}\sigma_s^2 T\right\}^{1-\frac{1}{\gamma}}\exp\left\{\frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2[(-\zeta_S+\sigma_s)^2+\zeta_\perp^2]T\right\}.\end{aligned} \tag{4.2.7}$$

To obtain the optimal terminal wealth, we insert the benchmark (4.2.1), the pricing kernel (2.3.12), and the expectation of the discounted benchmark (4.2.7) into the optimal wealth $\hat{X}_{T,S}$ in (4.2.2). This yields:

$$\begin{aligned}\hat{X}_{T,S}=&\exp\left\{\frac{1}{2}\zeta_S^2 T+\frac{1}{2}\zeta_\perp^2 T-\frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2[(-\zeta_S+\sigma_s)^2+\zeta_\perp^2]T\right\}\exp\left\{\int_0^T r_u du\right.\\ &\left.+\left(\frac{1}{\gamma}\zeta_S+\left(1-\frac{1}{\gamma}\right)\sigma_s\right)W^{\mathbb{P}}_{T,S}+\frac{1}{\gamma}\zeta_\perp W^{\mathbb{P}}_{T,\perp}\right\},\end{aligned} \tag{4.2.8}$$

where $\hat{X}_{T,S}$ is the optimal terminal wealth under the original measure $\mathbb{P}$. By the Girsanov theorem, we can change the Brownian motions in (4.2.8) from $\mathbb{P}$ to $\mathbb{Q}$. Thus, we obtain the optimal terminal wealth $\hat{X}_{T,S}$ under the risk-neutral measure $\mathbb{Q}$ as follows:

$$\begin{aligned}&\hat{X}_{T,S}\\ &=\exp\left\{\left(\frac{1}{2}-\frac{1}{\gamma}\right)\left(\zeta_S^2+\zeta_\perp^2\right)T-\frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2[(-\zeta_S+\sigma_s)^2+\zeta_\perp^2]T\right.\\ &\quad\left.+\left(1-\frac{1}{\gamma}\right)\sigma_s\zeta_S T\right\}\exp\left\{\int_0^T r_u du+\left(\frac{1}{\gamma}\zeta_S+\left(1-\frac{1}{\gamma}\right)\sigma_s\right)W^{\mathbb{Q}}_{T,S}+\frac{1}{\gamma}\zeta_\perp W^{\mathbb{Q}}_{T,\perp}\right\}.\end{aligned} \tag{4.2.9}$$

To simplify calculations, let C_T denote the deterministic term in (4.2.9). Then, we can rewrite the optimal terminal wealth $\hat{X}_{T,S}$ as follows:

$$\hat{X}_{T,S} = C_T \exp\left\{ \int_0^T r_u du + \left(\frac{1}{\gamma}\zeta_S + \left(1 - \frac{1}{\gamma}\right)\sigma_s\right) W^{\mathbb{Q}}_{T,S} + \frac{1}{\gamma}\zeta_\perp W^{\mathbb{Q}}_{T,\perp} \right\}. \tag{4.2.10}$$

After obtaining the optimal terminal wealth (4.2.10) under the stock benchmark by the martingale method, we proceed to compute the optimal wealth at an arbitrage time t.

Similar to the method used to derive the optimal wealth $\frac{\hat{X}_{t,C}}{B_t}$ under the constant benchmark in (4.1.18), we know that the discounted wealth $\frac{\hat{X}_{t,S}}{B_t}$ is equal to the conditional expectation of the discounted terminal wealth $\frac{\hat{X}_{T,S}}{B_T}$ under the $\mathbb{Q}$ measure. That is,

$$\frac{\hat{X}_{t,S}}{B_t} = \mathbb{E}^{\mathbb{Q}}\left[\frac{\hat{X}_{T,S}}{B_T}\middle|\mathscr{F}_t\right], \tag{4.2.11}$$

where the discounted optimal terminal wealth is

$$\frac{\hat{X}_{T,S}}{B_T} = C_T \exp\left\{ \left(\frac{1}{\gamma}\zeta_S + \left(1 - \frac{1}{\gamma}\right)\sigma_s\right) W^{\mathbb{Q}}_{T,S} + \frac{1}{\gamma}\zeta_\perp W^{\mathbb{Q}}_{T,\perp} \right\}. \tag{4.2.12}$$

To derive the optimal wealth $\frac{\hat{X}_{t,S}}{B_t}$, we compute the conditional expectation of $\frac{\hat{X}_{T,S}}{B_T}$ in (4.2.11) as follows:

$$\begin{aligned}\frac{\hat{X}_{t,S}}{B_t} = C_T \exp&\left\{ \left(\frac{1}{\gamma}\zeta_S + \left(1 - \frac{1}{\gamma}\right)\sigma_s\right) W^{\mathbb{Q}}_{t,S} + \frac{1}{\gamma}\zeta_\perp W^{\mathbb{Q}}_{t,\perp} \right\}\mathbb{E}^{\mathbb{Q}}\Bigg[\exp\Bigg\{\Bigg(\frac{1}{\gamma}\zeta_S \\ &+ \left(1 - \frac{1}{\gamma}\right)\sigma_s\Bigg)(W^{\mathbb{Q}}_{T,S} - W^{\mathbb{Q}}_{t,S}) + \frac{1}{\gamma}\zeta_\perp(W^{\mathbb{Q}}_{T,\perp} - W^{\mathbb{Q}}_{t,\perp})\Bigg\}\Bigg|\mathscr{F}_t\Bigg],\end{aligned} \tag{4.2.13}$$

where $W^{\mathbb{Q}}_{t,S}$ and $W^{\mathbb{Q}}_{t,\perp}$ are $\mathscr{F}_t$-measurable, which allows us to take these two Brownian motions out of the conditional expectation in (4.2.13). Similar to the way we computed the conditional expectation in (4.1.5), we consider the term

$$\left(\frac{1}{\gamma}\zeta_S + \left(1 - \frac{1}{\gamma}\right)\sigma_s\right)(W^{\mathbb{Q}}_{T,S} - W^{\mathbb{Q}}_{t,S}) + \frac{1}{\gamma}\zeta_\perp(W^{\mathbb{Q}}_{T,\perp} - W^{\mathbb{Q}}_{t,\perp}) \tag{4.2.14}$$

as a random variable. Then, the conditional expectation in (4.2.13) is the first order of the moment generating function of the variable (4.2.14). That is,

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}}\left[\exp\left\{\left(\frac{1}{\gamma}\zeta_S + \left(1 - \frac{1}{\gamma}\right)\sigma_s\right)(W^{\mathbb{Q}}_{T,S} - W^{\mathbb{Q}}_{t,S}) + \frac{1}{\gamma}\zeta_\perp(W^{\mathbb{Q}}_{T,\perp} - W^{\mathbb{Q}}_{t,\perp})\right\}\middle|\mathscr{F}_t\right] \\ &= \exp\left\{\frac{1}{2}\left[\left(\frac{1}{\gamma}\zeta_S + \frac{\gamma-1}{\gamma}\sigma_s\right)^2 + \frac{1}{\gamma^2}\zeta_\perp^2\right](T-t)\right\}.\end{aligned} \tag{4.2.15}$$

Insert the conditional expectation (4.2.15) into the optimal wealth in (4.2.13), and we obtain

$$\begin{aligned}\frac{\hat{X}_{t,S}}{B_t} = C_T \exp\Bigg\{\frac{1}{2}\Bigg[\left(\frac{1}{\gamma}\zeta_S + \frac{\gamma-1}{\gamma}\sigma_s\right)^2 \\ + \frac{1}{\gamma^2}\zeta_\perp^2\Bigg](T-t)\Bigg\} \exp\left\{\left(\frac{1}{\gamma}\zeta_S + \frac{\gamma-1}{\gamma}\sigma_s\right)W_{t,S}^{\mathbb{Q}} + \frac{1}{\gamma}\zeta_\perp W_{t,\perp}^{\mathbb{Q}}\right\}.\end{aligned} \tag{4.2.16}$$

Equation (4.2.16) implies that investors will not invest in stocks when the condition

$$\frac{1}{\gamma}\zeta_S + \frac{\gamma-1}{\gamma}\sigma_s = 0 \tag{4.2.17}$$

holds. Furthermore, they will not invest in zero-coupon bonds when $\zeta_\perp$ is zero. Let us compare the results of optimal wealth under the cash benchmark (4.1.26) and the stock benchmark (4.2.16). Observe that the amount of risk that investors willing to take for zero-coupon bonds are the same as coefficient terms $\frac{1}{\gamma}\zeta_\perp$ before Brownian motion $W_{t,\perp}^{\mathbb{Q}}$ are equal. Furthermore, investors are willing to bear more risks for stocks under the stock benchmark than the cash benchmark.

4.3 Incomplete Markets

In previous sections, we discussed the complete market case where the portfolio contains three different assets including stocks, money market accounts, and zero-coupon bonds. In incomplete markets, there exist risks that cannot be hedged, such as mortality risks, inflation risks, etc. Suppose that the market is incomplete now. According to the Second Fundamental Theorem of Asset Pricing, see Theorem 6, there exist an infinite number of equivalent martingale measures in incomplete markets. In this case, pricing kernels and discounted prices under equivalent martingale measures are also not unique.

Let us now include a new asset inflation-indexed bond I_t in the portfolio. If we assume that inflation-indexed bonds are traded in the market, then investors can extract the unique Radon-Nikodym derivative from the market price of inflation-indexed bonds, and then obtain the unique pricing kernel to discount the payoff. In this case, the market becomes complete, and we can still use the Martingale method to find optimal wealth and optimal hedging strategies.

Chen et al. (2017) proposed a model for inflation-indexed bonds where real interest rates and the expected inflation rates are assumed to be constant. We extend the Chen et al. (2017) model with a stochastic interest rate r_t following the Vasicek short rate model introduced in (2.1.5). The dynamics of the inflation-indexed bond is as follows:

$$dI_t = (r_t + \mu)I_t dt + \sigma_i I_t dW^{\mathbb{P}}_{t,I}, \tag{4.3.1}$$

where μ denotes the expected inflation rate, and σ_i denotes the volatility of the inflation-indexed bond. The unhedgeable risk in the incomplete markets is captured by the Brownian motion $W^{\mathbb{P}}_{t,I}$. Note that $W^{\mathbb{P}}_{t,I}$ is defined on the same probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, \mathbb{P})$ as $W^{\mathbb{P}}_{t,S}$ and $W^{\mathbb{P}}_{t,r}$. The sum of the real interest rate r_t and the expected inflation rate μ is equal to the nominal interest rate in the market. As the bond (4.3.1) follows a geometric Brownian motion, the solution of I_t is given by:

$$I_t = I_0 \exp\left\{\left(\mu + \frac{1}{2}\sigma_i^2\right)t + \int_0^t r_u du + \sigma_i W^{\mathbb{P}}_{t,I}\right\}. \tag{4.3.2}$$

Assume that the Brownian motion $W^{\mathbb{P}}_{t,I}$ and $W^{\mathbb{P}}_{t,r}$ are correlated with the correlation parameter $\rho_{r,I} \in (-1, 1)$. This implies,

$$dW^{\mathbb{P}}_{t,I} dW^{\mathbb{P}}_{t,r} = \rho_{r,I} dt. \tag{4.3.3}$$

Let $W^{\mathbb{P}}_{t,\top}$ be a Brownian motion that is independent of $W^{\mathbb{P}}_{t,r}$, i.e. $W^{\mathbb{P}}_{t,I} \perp\!\!\!\perp W^{\mathbb{P}}_{t,\top}$. Hence, we can use the factor model approach by Primbs (2016) to rewrite the Brownian motion $W^{\mathbb{P}}_{t,I}$ as follows:

$$dW^{\mathbb{P}}_{t,I} = \rho_{r,I} dW^{\mathbb{P}}_{t,r} + \sqrt{1 - \rho_{r,I}^2}\, dW^{\mathbb{P}}_{t,\top}. \tag{4.3.4}$$

Note that $W^{\mathbb{P}}_{t,r}$ and $W^{\mathbb{P}}_{t,S}$ are also correlated, and the correlation is shown in (2.1.8). Substitute the Brownian motion $W^{\mathbb{P}}_{t,r}$ in (4.3.4) with (2.1.8), and we can rewrite the Brownian motion $W^{\mathbb{P}}_{t,I}$ as a factor model by Primbs (2016) as follows:

$$dW^{\mathbb{P}}_{t,I} = \rho_{r,I}\left(\rho_{r,S} dW^{\mathbb{P}}_{t,S} + \sqrt{1 - \rho_{r,S}^2}\, dW^{\mathbb{P}}_{t,\perp}\right) + \sqrt{1 - \rho_{r,I}^2}\, dW^{\mathbb{P}}_{t,\top}, \tag{4.3.5}$$

where $W^{\mathbb{P}}_{t,I}$ is spanned by three Brownian motions $W^{\mathbb{P}}_{t,S}$, $W^{\mathbb{P}}_{t,\perp}$, and $W^{\mathbb{P}}_{t,\top}$ that are independent with each other. Equation (4.3.5) implies that these three independent Brownian motions are all correlated with $W^{\mathbb{P}}_{t,I}$. That is,

$$dW^{\mathbb{P}}_{t,I} dW^{\mathbb{P}}_{t,S} = \rho_{r,I}\rho_{r,S} dt, \tag{4.3.6}$$

$$dW^{\mathbb{P}}_{t,I} dW^{\mathbb{P}}_{t,\perp} = \rho_{r,I}\sqrt{1 - \rho_{r,S}^2}\, dt, \tag{4.3.7}$$

and,

$$dW^{\mathbb{P}}_{t,I} dW^{\mathbb{P}}_{t,\top} = \sqrt{1-\rho^2_{r,I}}dt. \tag{4.3.8}$$

Note that the inflation-indexed bond I_t is under the original measure $\mathbb{P}$ in (4.3.1). To obtain the bond process under the measure $\mathbb{Q}$, we first divide the process I_t by the numeraire B_t. The discounted bond process is as follows:

$$\begin{aligned} d\frac{I_t}{B_t} &= \mu\frac{I_t}{B_t}dt + \sigma_i\frac{I_t}{B_t}dW^{\mathbb{P}}_{t,I} \\ &= \mu\frac{I_t}{B_t}dt + \sigma_i\left[\rho_{r,I}\left(\rho_{r,S}\frac{I_t}{B_t}dW^{\mathbb{P}}_{t,S} + \sqrt{1-\rho^2_{r,S}}\frac{I_t}{B_t}dW^{\mathbb{P}}_{t,\perp}\right) + \sqrt{1-\rho^2_{r,I}}\frac{I_t}{B_t}dW^{\mathbb{P}}_{t,\top}\right]. \end{aligned} \tag{4.3.9}$$

By the Girsanov theorem, the following property holds:

$$dW^{\mathbb{Q}}_{t,\top} = dW^{\mathbb{P}}_{t,\top} + \zeta_{t,\top}dt, \tag{4.3.10}$$

where $\zeta_{t,I}$ denotes the Radon-Nikodym exponent. As we include inflation-indexed bonds in the hedging strategy, the market is complete. Hence, pricing kernels M_t and the discounted prices M_tX_t under equivalent martingale measures are still unique. Here, $\zeta_{t,I}$ is uniquely defined. By the change of measure, we obtain the discounted bond process under $\mathbb{Q}$ as follows:

$$\begin{aligned} d\frac{I_t}{B_t} &= \frac{I_t}{B_t}\Bigg[\left(\mu - \sigma_i\left(\rho_{r,I}\left[\rho_{r,S}\zeta_S + \sqrt{1-\rho^2_{r,S}}\zeta_\perp\right] + \sqrt{1-\rho^2_{r,I}}\zeta_\top\right)\right)dt \\ &\quad + \sigma_i\left(\rho_{r,I}\left[\rho_{r,S}dW^{\mathbb{Q}}_{t,S} + \sqrt{1-\rho^2_{r,S}}dW^{\mathbb{Q}}_{t,\perp}\right] + \sqrt{1-\rho^2_{r,I}}dW^{\mathbb{Q}}_{t,\top}\right)\Bigg] \\ &= \frac{I_t}{B_t}\left[\sigma_i\left(\rho_{r,I}\left[\rho_{r,S}dW^{\mathbb{Q}}_{t,S} + \sqrt{1-\rho^2_{r,S}}dW^{\mathbb{Q}}_{t,\perp}\right] + \sqrt{1-\rho^2_{r,I}}dW^{\mathbb{Q}}_{t,\top}\right)\right], \end{aligned} \tag{4.3.11}$$

where $\frac{I_t}{B_t}$ is a martingale with a zero-drift term under $\mathbb{Q}$. Equation (4.3.11) implies that the Radon-Nikodym exponent is as follows:

$$\zeta_{t,\top} = \frac{1}{\sqrt{1-\rho^2_{r,I}}}\left(\frac{\mu}{\sigma_i} - \rho_{r,I}\left[\rho_{r,S}\zeta_S + \sqrt{1-\rho^2_{r,S}}\zeta_\perp\right]\right). \tag{4.3.12}$$

Insert the Radon-Nikodym exponent $\zeta_{t,\perp}$ (2.3.11) and $\zeta_{t,S}$ (2.2.6) into (4.3.13), and we obtain

$$\zeta_{t,\top} = \frac{1}{\sqrt{1-\rho^2_{r,I}}}\left(\frac{\mu}{\sigma_i} - \frac{\rho_{r,I}(\bar{r}^* - \bar{r})\kappa}{\sigma_r}\right). \tag{4.3.13}$$

Note that $\zeta_{t,\top}$ is time-independent. Thus, we change the notion of the Radon-Nikodym exponent $\zeta_{t,I}$ into ζ_I in the remainder of this paper. Under the original measure $\mathbb{P}$, the new pricing kernel is given by

$$M_{T,I} = \exp\bigg\{ -\int_0^T r_u du - \frac{1}{2}\zeta_S^2 T - \frac{1}{2}\zeta_\perp^2 T - \frac{1}{2}\zeta_\top^2 T - \zeta_S W_{T,S}^{\mathbb{P}} - \zeta_\perp W_{T,\perp}^{\mathbb{P}} - \zeta_I W_{T,\top}^{\mathbb{P}} \bigg\}, \tag{4.3.14}$$

where we include the new Brownian motion $W_{T,\top}^{\mathbb{P}}$ in the previous pricing kernel (2.3.12).

After including the new asset I_t in the portfolio, we need to extend the wealth process (2.4.10) with the dynamics of inflation-indexed bonds (4.3.5). Let $\phi = \{\phi_{t,S}, \phi_{t,B}, \phi_{t,P}, \phi_{t,I}\}$ be the new set of hedging strategies. Then, we obtain the new wealth process X_t as follows:

$$dX_t = \phi_{t,S} dS_t + \phi_{t,B} dB_t + \phi_{t,P} dP_{(t,T)} + \phi_{t,I} dI_t. \tag{4.3.15}$$

Equivalently,

$$X_t = X_0 + \int_0^t \phi_{u,S} dS_u + \int_0^t \phi_{u,B} dB_u + \int_0^t \phi_{u,P} dP_{(u,T)} + \int_0^t \phi_{u,I} dI_u. \tag{4.3.16}$$

If we divide the wealth process X_t by the numeraire B_t, then we obtain the discounted wealth process as follows:

$$\frac{X_t}{B_t} = \frac{X_0}{B_0} + \int_0^t \phi_{u,S} d\frac{S_u}{B_u} + \int_0^t \phi_{u,P} d\frac{P_{(u,T)}}{B_u} + \int_0^t \phi_{u,I} d\frac{I_u}{B_u}, \tag{4.3.17}$$

where $\frac{X_t}{B_t}$ is a martingale under the measure $\mathbb{Q}$.

4.4 Optimal Wealth under an Inflation-Indexed Bond Benchmark

In section 4.1 and 4.2, we showed the optimal wealth under a constant and a stock benchmark respectively. Let us now analyst the optimal wealth when the benchmark $Y_{T,I}$ follows the inflation-indexed bond process in (4.3.2). That is,

$$Y_{T,I} = I_0 \exp\bigg\{ \Big(\mu + \frac{1}{2}\sigma_i^2\Big) T + \int_0^T r_u du + \sigma_i W_{T,I}^{\mathbb{P}} \bigg\}. \tag{4.4.1}$$

Equivalently,

$$
\begin{aligned}
Y_{T,I} = I_0 \exp\bigg\{ &\Big(\mu + \frac{1}{2}\sigma_i^2\Big)T + \int_0^T r_u du \\
&+ \sigma_i \bigg(\rho_{r,I}\Big[\rho_{r,S} dW_{t,S}^{\mathbb{P}} + \sqrt{1-\rho_{r,S}^2} dW_{t,\perp}^{\mathbb{P}}\Big] + \sqrt{1-\rho_{r,I}^2} dW_{t,\top}^{\mathbb{P}} \bigg) \bigg\}.
\end{aligned} \tag{4.4.2}
$$

As we derived the general solution of the optimal terminal wealth in (3.3.8), the optimal terminal wealth under the benchmark $Y_{T,I}$ is as follows:

$$
\hat{X}_{T,I} = \frac{M_{T,I}^{-\frac{1}{\gamma}} Y_{T,I}^{1-\frac{1}{\gamma}}}{\mathbb{E}[(M_{T,I}Y_{T,I})^{1-\frac{1}{\gamma}}]}, \tag{4.4.3}
$$

where the pricing kernel $M_{T,I}$ is shown in (4.3.14). To derive the optimal terminal wealth $\hat{X}_{T,I}$, we first compute the expectation term in (4.4.3) as follows:

$$
\begin{aligned}
&\mathbb{E}[(M_{T,I}Y_{T,I})^{1-\frac{1}{\gamma}}] \\
&= \exp\bigg\{ -\frac{1}{2}\zeta_S^2 T - \frac{1}{2}\zeta_\perp^2 T - \frac{1}{2}\zeta_\top^2 T + \mu T + \frac{1}{2}\sigma_i^2 T \bigg\}^{1-\frac{1}{\gamma}} \mathbb{E}\bigg[\exp\bigg\{ (\sigma_i\rho_{r,I}\rho_{r,S} - \zeta_S) W_{T,S}^{\mathbb{P}} \\
&\quad + \Big(\sigma_i\rho_{r,I}\sqrt{1-\rho_{r,S}^2} - \zeta_\perp\Big) W_{T,\perp}^{\mathbb{P}} + \Big(\sigma_i\sqrt{1-\rho_{r,I}^2} - \zeta_\top\Big) W_{T,\top}^{\mathbb{P}} \bigg\}^{1-\frac{1}{\gamma}} \bigg].
\end{aligned} \tag{4.4.4}
$$

Let the term

$$
(\sigma_i\rho_{r,I}\rho_{r,S} - \zeta_S) W_{T,S}^{\mathbb{P}} \Big(\sigma_i\rho_{r,I}\sqrt{1-\rho_{r,S}^2} - \zeta_\perp\Big) W_{T,\perp}^{\mathbb{P}} + \Big(\sigma_i\sqrt{1-\rho_{r,I}^2} - \zeta_\top\Big) W_{T,\top}^{\mathbb{P}} \tag{4.4.5}
$$

in (4.4.4) be a random variable. Then, the expectation term in (4.4.4) is the first order of the moment generating function of this random variable. Hence, we obtain

$$
\begin{aligned}
&\mathbb{E}\bigg[\exp\bigg\{ (\sigma_i\rho_{r,I}\rho_{r,S} - \zeta_S) W_{T,S}^{\mathbb{P}} + \Big(\sigma_i\rho_{r,I}\sqrt{1-\rho_{r,S}^2} - \zeta_\perp\Big) W_{T,\perp}^{\mathbb{P}} \\
&\qquad + \Big(\sigma_i\sqrt{1-\rho_{r,I}^2} - \zeta_\top\Big) W_{T,\top}^{\mathbb{P}} \bigg\}^{1-\frac{1}{\gamma}} \bigg] \\
&= \exp\bigg\{ \frac{1}{2}\Big(1-\frac{1}{\gamma}\Big)^2 \bigg[(\sigma_i\rho_{r,I}\rho_{r,S} - \zeta_S)^2 + \Big(\sigma_i\rho_{r,I}\sqrt{1-\rho_{r,S}^2} - \zeta_\perp\Big)^2 \\
&\qquad + \Big(\sigma_i\sqrt{1-\rho_{r,I}^2} - \zeta_\top\Big)^2 \bigg] T \bigg\},
\end{aligned} \tag{4.4.6}
$$

where the Brownian motion $W^{\mathbb{P}}_{T,S}$, $W^{\mathbb{P}}_{T,\perp}$, and $W^{\mathbb{P}}_{T,\top}$ are not correlated. Insert (4.4.6) into (4.4.4), and we obtain the expectation of discounted benchmark as follows:

$$\begin{aligned}
&\mathbb{E}[(M_{T,I}Y_{T,I})^{1-\frac{1}{\gamma}}] \\
&= \exp\bigg\{-\frac{1}{2}\zeta_S^2 T - \frac{1}{2}\zeta_\perp^2 T - \frac{1}{2}\zeta_\top^2 T + \mu T + \frac{1}{2}\sigma_i^2 T\bigg\}^{1-\frac{1}{\gamma}} \exp\bigg\{\frac{1}{2}\Big(1-\frac{1}{\gamma}\Big)^2 \\
&\qquad \bigg[(\sigma_i\rho_{r,I}\rho_{r,S}-\zeta_S)^2 + \Big(\sigma_i\rho_{r,I}\sqrt{1-\rho_{r,S}^2}-\zeta_\perp\Big)^2 + \Big(\sigma_i\sqrt{1-\rho_{r,I}^2}-\zeta_\top\Big)^2\bigg]T\bigg\}.
\end{aligned} \tag{4.4.7}$$

Now we insert the benchmark (4.4.2), the pricing kernel (4.3.14), and the expectation of the discounted benchmark (5.3.3) into the optimal terminal wealth (4.4.3) . Thus, we obtain the optimal terminal wealth $\hat{X}_{T,I}$ as follows:

$$\begin{aligned}
\hat{X}_{T,I} = \exp\bigg\{&\frac{1}{2}\zeta_S^2 T + \frac{1}{2}\zeta_\perp^2 T + \frac{1}{2}\zeta_I^2 T + \frac{1}{2}\Big(1-\frac{1}{\gamma}\Big)^2\bigg[(\sigma_i\rho_{r,I}\rho_{r,S}-\zeta_S)^2 \\
&+ \Big(\sigma_i\rho_{r,I}\sqrt{1-\rho_{r,S}^2}-\zeta_\perp\Big)^2 + \Big(\sigma_i\sqrt{1-\rho_{r,I}^2}-\zeta_\top\Big)^2\bigg]T\bigg\}\exp\bigg\{\int_0^T r_u du \\
&+ \Big(\frac{1}{\gamma}\zeta_S + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\rho_{r,S}\Big)W^{\mathbb{P}}_{T,S} + \Big(\frac{1}{\gamma}\zeta_\perp + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\sqrt{1-\rho_{r,S}^2}\Big)W^{\mathbb{P}}_{T,\perp} \\
&+ \Big(\frac{1}{\gamma}\zeta_\top + \frac{\gamma-1}{\gamma}\sigma_i\sqrt{1-\rho_{r,I}^2}\Big)W^{\mathbb{P}}_{T,\top}\bigg\}.
\end{aligned} \tag{4.4.8}$$

To simplify calculations, let O_T denote the deterministic term in (4.4.8). Thus, we can rewrite the optimal terminal wealth (4.4.8) as follows:

$$\begin{aligned}
\hat{X}_{T,I} = O_T\exp\bigg\{&\int_0^T r_u du + \Big(\frac{1}{\gamma}\zeta_S + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\rho_{r,S}\Big)W^{\mathbb{P}}_{T,S} + \Big(\frac{1}{\gamma}\zeta_\perp \\
&+ \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\sqrt{1-\rho_{r,S}^2}\Big)W^{\mathbb{P}}_{T,\perp} + \Big(\frac{1}{\gamma}\zeta_\top + \frac{\gamma-1}{\gamma}\sigma_i\sqrt{1-\rho_{r,I}^2}\Big)W^{\mathbb{P}}_{T,\top}\bigg\}.
\end{aligned} \tag{4.4.9}$$

Note that the optimal terminal wealth (4.4.9) is under the original measure $\mathbb{P}$. By the Girsanov theorem, the Brownian motions in (4.4.9) can be changed to the equivalent martingale measure $\mathbb{Q}$. Thus, we obtain:

$$
\begin{aligned}
&\hat{X}_{T,I} \\
&= O_T \exp\bigg\{ -\left(\frac{1}{\gamma}\zeta_S + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\rho_{r,S}\right)\zeta_S T - \left(\frac{1}{\gamma}\zeta_\perp + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\sqrt{1-\rho_{r,S}^2}\right)\zeta_\perp T \\
&\quad - \left(\frac{1}{\gamma}\zeta_\top + \frac{\gamma-1}{\gamma}\sigma_i\sqrt{1-\rho_{r,I}^2}\right)\zeta_\top T\bigg\} \exp\bigg\{ \int_0^T r_u du + \left(\frac{1}{\gamma}\zeta_S\right. \\
&\quad \left. + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\rho_{r,S}\right) W_{T,S}^{\mathbb{Q}} + \left(\frac{1}{\gamma}\zeta_\perp + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\sqrt{1-\rho_{r,S}^2}\right) W_{T,\perp}^{\mathbb{Q}} \\
&\quad + \left(\frac{1}{\gamma}\zeta_\top + \frac{\gamma-1}{\gamma}\sigma_i\sqrt{1-\rho_{r,I}^2}\right) W_{T,\top}^{\mathbb{Q}}\bigg\},
\end{aligned}
\tag{4.4.10}
$$

where the optimal terminal wealth (4.4.10) is under the measure $\mathbb{Q}$ now. To facilitate calculations, let $O_T^{'}$ denote the deterministic term in (4.4.10). That is,

$$
\begin{aligned}
O_T^{'} = \exp\bigg\{ &\frac{1}{2}\zeta_S^2 T + \frac{1}{2}\zeta_\perp^2 T + \frac{1}{2}\zeta_I^2 T + \frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2 \bigg[(\sigma_i\rho_{r,I}\rho_{r,S} - \zeta_S)^2 \\
&+ \left(\sigma_i\rho_{r,I}\sqrt{1-\rho_{r,S}^2} - \zeta_\perp\right)^2 + \left(\sigma_i\sqrt{1-\rho_{r,I}^2} - \zeta_\top\right)^2\bigg] T \\
&- \left(\frac{1}{\gamma}\zeta_S + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\rho_{r,S}\right)\zeta_S T - \left(\frac{1}{\gamma}\zeta_\perp + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\sqrt{1-\rho_{r,S}^2}\right)\zeta_\perp T \\
&- \left(\frac{1}{\gamma}\zeta_\top + \frac{\gamma-1}{\gamma}\sigma_i\sqrt{1-\rho_{r,I}^2}\right)\zeta_\top T\bigg\}.
\end{aligned}
\tag{4.4.11}
$$

Now we can rewrite the optimal terminal wealth in (4.4.10) as follows:

$$
\begin{aligned}
\hat{X}_{T,I} = O_T^{'} \exp\bigg\{ &\int_0^T r_u du + \left(\frac{1}{\gamma}\zeta_S + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\rho_{r,S}\right) W_{T,S}^{\mathbb{Q}} + \left(\frac{1}{\gamma}\zeta_\perp\right. \\
&\left. + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\sqrt{1-\rho_{r,S}^2}\right) W_{T,\perp}^{\mathbb{Q}} + \left(\frac{1}{\gamma}\zeta_\top + \frac{\gamma-1}{\gamma}\sigma_i\sqrt{1-\rho_{r,I}^2}\right) W_{T,\top}^{\mathbb{Q}}\bigg\}.
\end{aligned}
\tag{4.4.12}
$$

As we obtained the optimal terminal wealth (4.4.12) under the inflation-indexed benchmark $Y_{T,I}$, we now proceed to derive the optimal wealth $X_{t,I}$ at an arbitrage time t. We first compute the discounted optimal wealth as follows:

$$\frac{\hat{X}_{T,I}}{B_T} = O_T' \exp\left\{\left(\frac{1}{\gamma}\zeta_S + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\rho_{r,S}\right)W^{\mathbb{Q}}_{T,S} + \left(\frac{1}{\gamma}\zeta_\perp \right.\right.$$
$$\left.\left. + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\sqrt{1-\rho^2_{r,S}}\right)W^{\mathbb{Q}}_{T,\perp} + \left(\frac{1}{\gamma}\zeta_\top + \frac{\gamma-1}{\gamma}\sigma_i\sqrt{1-\rho^2_{r,I}}\right)W^{\mathbb{Q}}_{T,\top}\right\}. \tag{4.4.13}$$

The discounted wealth $\frac{\hat{X}_{t,I}}{B_t}$ at an arbitrage time t equals to the conditional expectation of the discounted terminal wealth $\frac{\hat{X}_{T,I}}{B_T}$ given the filtration $\mathscr{F}_t$. That is,

$$\frac{\hat{X}_{t,I}}{B_t} = \mathbb{E}^{\mathbb{Q}}\left[\frac{\hat{X}_{T,I}}{B_T}\middle|\mathscr{F}_t\right]. \tag{4.4.14}$$

Insert the discounted terminal wealth (4.4.13) into (4.4.14), and we obtain the discounted optimal wealth as follows:

$$\frac{\hat{X}_{t,I}}{B_t} = O_T' \exp\left\{\left(\frac{1}{\gamma}\zeta_S + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\rho_{r,S}\right)W^{\mathbb{Q}}_{t,S} + \left(\frac{1}{\gamma}\zeta_\perp + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\sqrt{1-\rho^2_{r,S}}\right)W^{\mathbb{Q}}_{t,\perp}\right.$$
$$\left. + \left(\frac{1}{\gamma}\zeta_\top + \frac{\gamma-1}{\gamma}\sigma_i\sqrt{1-\rho^2_{r,I}}\right)W^{\mathbb{Q}}_{t,\top}\right\}\mathbb{E}^{\mathbb{Q}}\left[\exp\left\{\left(\frac{1}{\gamma}\zeta_S\right.\right.\right.$$
$$\left. + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\rho_{r,S}\right)(W^{\mathbb{Q}}_{T,S} - W^{\mathbb{Q}}_{t,S}) + \left(\frac{1}{\gamma}\zeta_\perp + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\sqrt{1-\rho^2_{r,S}}\right)(W^{\mathbb{Q}}_{T,\perp}$$
$$\left.\left.\left. - W^{\mathbb{Q}}_{t,\perp}) + \left(\frac{1}{\gamma}\zeta_\top + \frac{\gamma-1}{\gamma}\sigma_i\sqrt{1-\rho^2_{r,I}}\right)(W^{\mathbb{Q}}_{T,\top} - W^{\mathbb{Q}}_{t,\top})\right\}\middle|\mathscr{F}_t\right]. \tag{4.4.15}$$

Note that the Brownian motion $W^{\mathbb{Q}}_{t,S}$, $W^{\mathbb{Q}}_{t,\perp}$, and $W^{\mathbb{Q}}_{t,\top}$ are $\mathscr{F}_t$-measurable. Thus, we can take these Brownian motion out of the conditional expectation in (4.4.15). The expectation term in (4.4.15) is calculated as follows:

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left\{\left(\frac{1}{\gamma}\zeta_S + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\rho_{r,S}\right)(W^{\mathbb{Q}}_{T,S} - W^{\mathbb{Q}}_{t,S}) + \left(\frac{1}{\gamma}\zeta_\perp + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\sqrt{1-\rho^2_{r,S}}\right)\right.\right.$$
$$\left.\left.(W^{\mathbb{Q}}_{T,\perp} - W^{\mathbb{Q}}_{t,\perp}) + \left(\frac{1}{\gamma}\zeta_\top + \frac{\gamma-1}{\gamma}\sigma_i\sqrt{1-\rho^2_{r,I}}\right)(W^{\mathbb{Q}}_{T,\top} - W^{\mathbb{Q}}_{t,\top})\right\}\middle|\mathscr{F}_t\right]$$
$$= \exp\left\{\frac{1}{2}\left[\left(\frac{1}{\gamma}\zeta_S + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\rho_{r,S}\right)^2 + \left(\frac{1}{\gamma}\zeta_\perp + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\sqrt{1-\rho^2_{r,S}}\right)^2\right.\right.$$
$$\left.\left. + \left(\frac{1}{\gamma}\zeta_\top + \frac{\gamma-1}{\gamma}\sigma_i\sqrt{1-\rho^2_{r,I}}\right)^2\right](T-t)\right\}. \tag{4.4.16}$$

Inserting the conditional expectation (4.4.16) into (4.4.15), we obtain:

$$\begin{aligned}\frac{\hat{X}_{t,I}}{B_t} = O_T' \exp\Bigg\{\frac{1}{2}\Bigg[\left(\frac{1}{\gamma}\zeta_S + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\rho_{r,S}\right)^2 + \left(\frac{1}{\gamma}\zeta_\perp + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\sqrt{1-\rho_{r,S}^2}\right)^2 \\ + \left(\frac{1}{\gamma}\zeta_\top + \frac{\gamma-1}{\gamma}\sigma_i\sqrt{1-\rho_{r,I}^2}\right)^2\Bigg](T-t)\Bigg\} \exp\Bigg\{\left(\frac{1}{\gamma}\zeta_S + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\rho_{r,S}\right)W_{t,S}^{\mathbb{Q}} \\ + \left(\frac{1}{\gamma}\zeta_\perp + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\sqrt{1-\rho_{r,S}^2}\right)W_{t,\perp}^{\mathbb{Q}} + \left(\frac{1}{\gamma}\zeta_\top + \frac{\gamma-1}{\gamma}\sigma_i\sqrt{1-\rho_{r,I}^2}\right)W_{t,\top}^{\mathbb{Q}}\Bigg\},\end{aligned} \tag{4.4.17}$$

where the optimal wealth $\frac{\hat{X}_{t,I}}{B_t}$ is a martingale under $\mathbb{Q}$. Equation (4.4.17) implies that if the inequality

$$\zeta_\top \neq (1-\gamma)\sigma_i\sqrt{1-\rho_{r,I}^2} \tag{4.4.18}$$

holds, i.e. the coefficient term of Brownian motion $W_{t,\top}^{\mathbb{Q}}$ is not zero, then inflation-indexed bonds can be hedged, which ensures that the market is complete. In this case, the optimal wealth (4.4.17) that we found by using the Martingale method is indeed the unique optimal wealth. If the condition in (4.4.18) is not satisfied, then the market is still incomplete, and equivalent martingale measure are not unique. We will discuss the case about hedging strategies in incomplete markets in the following chapter.

Chapter 5

Optimal Hedging Strategies

In the previous chapter, we derived the optimal wealth under three benchmarks. This chapter presents the way to compute the optimal hedging strategies from the optimal wealth in both complete and incomplete markets.

5.1 Optimal Hedging under a Cash Benchmark

In section 4.1, we derived the optimal discounted wealth $\frac{\hat{X}_{t,C}}{B_t}$ under the cash benchmark. This section presents the optimal hedging strategy under the benchmark $Y_{t,C}$. In complete markets, the equivalent martingale measure $\mathbb{Q}$ is unique. Thus, the optimal wealth and optimal hedging strategy are also unique. To obtain the optimal hedging, we adopt the approach of matching coefficients of the Brownian motions of the derivative of the optimal wealth.

Let us first compute the derivative of the optimal discounted wealth $\frac{\hat{X}_{t,C}}{B_t}$ (4.1.26) under the measure $\mathbb{Q}$ as follows:

$$d\frac{\hat{X}_{t,C}}{B_t} = \left[\frac{\partial \frac{\hat{X}_{t,C}}{B_t}}{\partial t} + [...]\right] + \frac{1}{\gamma}\zeta_S \frac{\hat{X}_{t,C}}{B_t} dW^{\mathbb{Q}}_{t,S} + \frac{1}{\gamma}\zeta_\perp \frac{\hat{X}_{t,C}}{B_t} dW^{\mathbb{Q}}_{t,\perp}. \tag{5.1.1}$$

Note that the only factor influencing the optimal hedging strategy is the coefficients of the Brownian motion $dW^{\mathbb{Q}}_{t,S}$ and $dW^{\mathbb{Q}}_{t,\perp}$. Hence, it is not necessary to compute the derivative of drift term of the optimal discounted wealth in (5.1.1).

The portfolio consists of three assets, including stocks, money market accounts, and zero-coupon bonds. Let $\phi^C = \{\phi^C_{t,S}, \phi^C_{t,B}, \phi^C_{u,P}\}$ be the set of hedging strategies under the constant benchmark $Y_{t,C}$. We defined the discounted wealth in (2.4.12).

That is,

$$\frac{\hat{X}_{t,C}}{B_t} = \frac{X_{0,C}}{B_0} + \int_0^t \phi_{u,S}^C d\frac{S_u}{B_u} + \int_0^t \phi_{u,P}^C d\frac{P_{(u,T)}}{B_u}, \tag{5.1.2}$$

where the dynamics of the discounted stock $\frac{S_u}{B_u}$ and the discounted bond $\frac{P_{(u,T)}}{B_u}$ under the risk-neutral measure $\mathbb{Q}$ are shown in (2.2.5) and (2.4.9) respectively. Let us now match the coefficients of the Brownian motion $dW_{t,S}^{\mathbb{Q}}$ between (5.1.1) and (5.1.2). We know that both the stock process and the bond process contain the Brownian $W_{t,S}^{\mathbb{Q}}$. Hence, we obtain:

$$\frac{1}{\gamma}\zeta_S\frac{\hat{X}_{t,C}}{B_t}dW_{t,S}^{\mathbb{Q}} = \phi_{t,S}^C\sigma_s\frac{S_u}{B_u} - \phi_{t,P}^C\sigma_r K_{(t,T)}\rho_{r,S}\frac{P_{(u,T)}}{B_u}dW_{t,S}^{\mathbb{Q}}, \tag{5.1.3}$$

where the left-hand side (LHS) of equation (5.1.3) is the Brownian motion term $dW_{t,S}^{\mathbb{Q}}$ in (5.1.1), and the right-hand side (RHS) is the $dW_{t,S}^{\mathbb{Q}}$ term in (5.1.2). Additionally, let us also match the coefficient of the Brownian motion $dW_{t,\perp}^{\mathbb{Q}}$. Note that only the zero-coupon bond process contains the Brownian $W_{t,\perp}^{\mathbb{Q}}$. Similar to the way we derive equation (5.1.3), we obtain:

$$\frac{1}{\gamma}\zeta_\perp\frac{\hat{X}_{t,C}}{B_t}dW_{t,\perp}^{\mathbb{Q}} = -\phi_{t,P}^C\sigma_r K_{(t,T)}\sqrt{1-\rho_{r,S}^2}\frac{P_{(u,T)}}{B_u}dW_{t,\perp}^{\mathbb{Q}}, \tag{5.1.4}$$

where the LHS and the RHS of (5.1.4) are the $dW_{t,\perp}^{\mathbb{Q}}$ term in the optimal discounted wealth process (5.1.1) and (5.1.2) respectively.

Combing equation (5.1.3) and (5.1.4), we obtain an equivalent matrix equation as follows:

$$\begin{bmatrix} \frac{1}{\gamma}\zeta_S\frac{\hat{X}_{t,C}}{B_t} \\ \frac{1}{\gamma}\zeta_\perp\frac{\hat{X}_{t,C}}{B_t} \end{bmatrix} = \begin{bmatrix} \sigma_s\frac{S_t}{B_t} & -\sigma_r K_{(t,T)}\rho_{r,S}\frac{P_{(t,T)}}{B_t} \\ 0 & -\sigma_r K_{(t,T)}\sqrt{1-\rho_{r,S}^2}\frac{P_{(t,T)}}{B_t} \end{bmatrix} \begin{bmatrix} \phi_{t,S}^C \\ \phi_{t,P}^C \end{bmatrix}. \tag{5.1.5}$$

Solve the above matrix equation (5.1.5), and it yields:

$$\phi_{t,S}^C = \hat{X}_{t,C}\frac{1}{S_t}\left[\frac{1}{\gamma}\frac{1}{1-\rho_{r,S}^2}\left(\frac{\lambda}{\sigma_s^2} - \frac{(\bar{r}^*-\bar{r})\kappa\rho_{r,S}}{\sigma_r\sigma_s}\right)\right], \tag{5.1.6}$$

and,

$$\phi_{t,P}^C = -\hat{X}_{t,C}\frac{1}{P_{(t,T)}}\left[\frac{1}{\gamma}\frac{1}{K_{(t,T)}}\frac{1}{1-\rho_{r,S}^2}\left(\frac{(\bar{r}^*-\bar{r})\kappa}{\sigma_r^2} - \frac{\rho_{r,S}\lambda}{\sigma_s\sigma_r}\right)\right], \tag{5.1.7}$$

where $\phi_{t,S}^C$ (5.1.6) and $\phi_{t,P}^C$ (5.1.7) are optimal hedging for stocks and zero-coupon bonds under the constant benchmark respectively. After obtaining the hedging strategy $\phi_{t,S}^C$ and $\phi_{t,P}^C$, investors can invest the remaining capital in the money market account. That is,

$$\phi_{t,B}^C = \hat{X}_{t,C} - \phi_{t,S}^C S_t - \phi_{u,P}^C P_{(u,T)}. \tag{5.1.8}$$

Equation (5.1.6) and (5.1.7) imply that the optimal hedging strategy $\phi_{t,S}^C$ and $\phi_{u,P}^C$ converge to 0 when γ, σ_r, $\sigma_s \to \infty$. This means that investors will only invest in the money market account if they are extremely risk-averse, or when the stock and the interest rate are extremely volatile. Observe that the correlation between stocks and bonds determines whether investor should hold long or short positions in stocks and bonds. From equation (5.1.6), we know that investor hold a long position in stocks, and a short position in bonds, i.e. $\phi_{t,S}^C > 0$ and $\phi_{u,P}^C < 0$, if the following condition is satisfied:

$$\rho_{r,S} < \frac{\lambda\sigma_r}{(\bar{r}^* - \bar{r})\kappa\sigma_s}. \tag{5.1.9}$$

Otherwise, investors hold a short position in stocks, and a long position in bonds.

5.2 Optimal hedging under a Stock Benchmark

In section 5.1, we derived the set of optimal hedging strategies under the constant benchmark. In this section, we use the same method to find the optimal hedging under the stock benchmark $Y_{t,S}$.

Similar to the computation of the dynamics of the optimal discounted wealth $\frac{\hat{X}_{t,C}}{B_t}$ under the constant benchmark, we first derive the derivative of $\frac{\hat{X}_{t,S}}{B_t}$ from the optimal discounted wealth in (4.2.16) as follows:

$$d\frac{\hat{X}_{t,S}}{B_t} = \left[\frac{\partial \frac{\hat{X}_{t,S}}{B_t}}{\partial t} + [...]\right] + \left(\frac{1}{\gamma}\zeta_S + \frac{\gamma-1}{\gamma}\sigma_s^2\right)\frac{\hat{X}_{t,S}}{B_t}dW_{t,S}^{\mathbb{Q}} + \frac{1}{\gamma}\zeta_\perp\frac{\hat{X}_{t,S}}{B_t}dW_{t,\perp}^{\mathbb{Q}}, \tag{5.2.1}$$

where the drift term in (5.2.1) is not computed as we only need to match the coefficients of the Brownian motions to obtain the optimal hedging strategies. Let $\phi^S = \{\phi_{t,S}^S, \phi_{t,B}^S, \phi_{u,P}^S\}$ denote the set of hedging strategies for stocks, money market accounts, and zero-coupon bonds respectively. The discounted wealth process $\frac{\hat{X}_{t,S}}{B_t}$ under the stock benchmark is the same as the one under the constant benchmark in (5.1.2). That is,

$$\frac{\hat{X}_{t,S}}{B_t} = \frac{X_{0,S}}{B_0} + \int_0^t \phi_{u,S}^S d\frac{S_u}{B_u} + \int_0^t \phi_{u,P}^S d\frac{P_{(u,T)}}{B_u}. \tag{5.2.2}$$

We match the coefficients of the Brownian motion $dW_{t,S}^{\mathbb{Q}}$ in the optimal discounted wealth in (5.2.1)and (5.2.2). It yields:

$$\left(\frac{1}{\gamma}\zeta_S + \frac{\gamma-1}{\gamma}\sigma_s^2\right)\frac{\hat{X}_{t,S}}{B_t}dW_{t,S}^{\mathbb{Q}} = \phi_{t,S}\sigma_s\frac{S_t}{B_t}dW_{t,S}^{\mathbb{Q}} - \phi_{t,P}\sigma_r K_{(t,T)}\rho_{r,S}\frac{P_{(t,T)}}{B_t}dW_{t,S}^{\mathbb{Q}}, \tag{5.2.3}$$

where the LHS and the RHS are the Brownian motion $dW_{t,S}^{\mathbb{Q}}$ term in the optimal discounted wealth in (5.2.1) and (5.2.2) respectively. Note that $dW_{t,S}^{\mathbb{Q}}$ is contained in both stock process and the bond process, but $dW_{t,\perp}^{\mathbb{Q}}$ only appears in the bond process. Similarly, we match the coefficients of the Brownian motion $dW_{t,\perp}^{\mathbb{Q}}$ in (5.2.1) and (5.2.2) as follows:

$$\frac{1}{\gamma}\zeta_\perp\frac{\hat{X}_{t,S}}{B_t}dW_{t,\perp}^{\mathbb{Q}} = -\phi_{t,P}\sigma_r K_{(t,T)}\sqrt{1-\rho_{r,S}^2}\frac{P_{(t,T)}}{B_t}dW_{t,\perp}^{\mathbb{Q}}. \tag{5.2.4}$$

Let us set up a matrix equation by combining equation (5.2.3) and (5.2.4) as follows:

$$\begin{bmatrix} \left(\frac{1}{\gamma}\zeta_S + \frac{\gamma-1}{\gamma}\sigma_s^2\right)\frac{\hat{X}_{t,S}}{B_t} \\ \frac{1}{\gamma}\zeta_\perp\frac{\hat{X}_{t,S}}{B_t} \end{bmatrix} = \begin{bmatrix} \sigma_s S_t & -\sigma_r K_{(t,T)}\rho_{r,S}\frac{P_{(t,T)}}{B_t} \\ 0 & -\sigma_r K_{(t,T)}\sqrt{1-\rho_{r,S}^2}\frac{P_{(t,T)}}{B_t} \end{bmatrix} \begin{bmatrix} \phi_{t,S}^S \\ \phi_{t,P}^S \end{bmatrix}. \tag{5.2.5}$$

Notice that the RHS of the matrix equation is the same as the one in (5.1.5) as the portfolio contains the same assets. Solve the above matrix equation (5.2.5), and we obtain the hedging strategies for stocks and zero-coupon bonds as follows:

$$\phi_{t,S}^S = \hat{X}_t\frac{1}{S_t}\left[\frac{1}{\gamma}\left(\frac{\lambda}{\sigma_s^2} - \frac{\rho_{r,S}}{1-\rho_{r,S}^2}\left(\frac{(\bar{r}^*-\bar{r})\kappa}{\sigma_r\sigma_s} - \frac{\rho_{r,S}\lambda}{\sigma_s^2}\right)\right) - \frac{1-\gamma}{\gamma}\sigma_s\right], \tag{5.2.6}$$

and,

$$\phi_{t,P}^S = -\hat{X}_t\frac{1}{P_{(t,T)}}\left[\frac{1}{\gamma}\frac{1}{K_{(t,T)}}\frac{1}{1-\rho_{r,S}^2}\left(\frac{(\bar{r}^*-\bar{r})\kappa}{\sigma_r^2} - \frac{\rho_{r,S}\lambda}{\sigma_s\sigma_r}\right)\right]. \tag{5.2.7}$$

Additionally, the hedging strategy for the money market account is

$$\phi_{t,B}^S = \hat{X}_{t,S} - \phi_{t,S}^S S_t - \phi_{u,P}^S P_{(u,T)}. \tag{5.2.8}$$

Equation (5.2.7) indicates that the optimal hedging strategy for stocks $\phi_{t,S}$ becomes proportional to the volatility of stocks σ_s, when investors are extremely risk-averse, i.e. $\gamma \to \infty$. In this case, investors are extremely worried about deviating from the stock benchmark, so they prefer to perfectly replicate the stock benchmark. Additionally, $\phi_{u,P} \to 0$ if $\gamma \to \infty$. In this case, investors will not invest in zero-coupon bonds, and the hedging strategy $\phi^S_{t,B}$ for money market accounts only depends on the amount left after investing in the stock market.

5.3 Optimal Hedging under an Inflation-Indexed Bond Benchmark

In incomplete markets, the prices of any assets are not unique as there exist multiple equivalent martingale measures. In section 4.4, we included inflation-indexed bonds in the market, and derived the optimal discounted wealth under the inflation-indexed bond benchmark by the Martingale method. As inflation-indexed bonds were included in the hedging strategy, the market is complete. However, when inflation-indexed bonds can not be hedged, the market remains incomplete. In this case, optimal hedging strategies are also not unique, and we need to find the worst-case martingale measure among all equivalent martingale measures. This section analyses the optimal hedging strategy under the inflation-indexed bond benchmark in both complete and incomplete markets.

Let us analyse the complete market case first. To obtain the optimal hedging strategy, we use the same approach as the ones in section 5.1 and 5.2. Let us now compute the derivative of the optimal discounted wealth under the measure $\mathbb{Q}$ as follows:

$$d\frac{\hat{X}_{t,I}}{B_t} = \left[\frac{\partial \frac{\hat{X}_{t,I}}{B_t}}{\partial t} + [...]\right] + \left(\frac{1}{\gamma}\zeta_S + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\rho_{r,S}\right)\frac{\hat{X}_{t,I}}{B_t}dW^{\mathbb{Q}}_{t,S} + \left(\frac{1}{\gamma}\zeta_\perp + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\sqrt{1-\rho^2_{r,S}}\right)\frac{\hat{X}_{t,I}}{B_t}dW^{\mathbb{Q}}_{t,\perp} + \left(\frac{1}{\gamma}\zeta_\top + \frac{\gamma-1}{\gamma}\sigma_i\sqrt{1-\rho^2_{r,I}}\right)\frac{\hat{X}_{t,I}}{B_t}dW^{\mathbb{Q}}_{t,\top}. \tag{5.3.1}$$

Note that the current portfolio includes an additional asset, the inflation-indexed bond. As the inflation risk for the inflation-indexed bond is unhedgeable, the market is incomplete. The discounted wealth process (5.3.1) is under three uncorrelated Brownian motions, including $W^{\mathbb{Q}}_{t,S}$, $W^{\mathbb{Q}}_{t,\perp}$, and $W^{\mathbb{Q}}_{t,\top}$. Let $\phi^I = \{\phi^I_{t,S}, \phi^I_{t,B}, \phi^I_{u,P}, \phi^I_{t,I}\}$ denote the set of hedging strategies for stocks, money market accounts, zero-coupon bonds, and inflation-indexed bonds respectively. We can write the optimal discounted wealth as follows:

$$\frac{\hat{X}_{t,I}}{B_t} = \frac{X_{0,I}}{B_0} + \int_0^t \phi^I_{u,S} d\frac{S_u}{B_u} + \int_0^t \phi^I_{u,P} d\frac{P_{(u,T)}}{B_u} + \int_0^t \phi^I_{u,I} d\frac{I_u}{B_u}, \tag{5.3.2}$$

where the discounted asset process $\frac{S_u}{B_u}$, $\frac{P_{(u,T)}}{B_u}$, and $\frac{I_u}{B_u}$ are martingales under the measure $\mathbb{Q}$. Let us first match the coefficients of the Brownian motion $dW^{\mathbb{Q}}_{t,S}$. Note that $dW^{\mathbb{Q}}_{t,S}$ is contained in the stock process, the zero-coupon bond process, and the inflation-indexed bond process. Hence, we obtain:

$$\begin{aligned}&\left(\frac{1}{\gamma}\zeta_S + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\rho_{r,S}\right)\frac{\hat{X}_{t,I}}{B_t}dW^{\mathbb{Q}}_{t,S}\\ &= \phi^I_{t,S}\sigma_s\frac{S_t}{B_t}dW^{\mathbb{Q}}_{t,S} - \phi^I_{t,P}\sigma_r K_{(t,T)}\rho_{r,S}\frac{P_{(t,T)}}{B_t}dW^{\mathbb{Q}}_{t,S} + \phi^I_{t,I}\sigma_i\rho_{r,I}\rho_{r,S}\frac{I_t}{B_t}dW^{\mathbb{Q}}_{t,S},\end{aligned} \tag{5.3.3}$$

where the LHS and the RHS of equation (5.3.3) are the $dW^{\mathbb{Q}}_{t,S}$ term in the optimal discounted wealth process (5.3.1) and (5.3.2) respectively. Similarly, we match the coefficients of the second Brownian motion $W^{\mathbb{Q}}_{t,\perp}$. This yields:

$$\begin{aligned}&\left(\frac{1}{\gamma}\zeta_\perp + \frac{\gamma-1}{\gamma}\sigma_i\rho_{r,I}\sqrt{1-\rho^2_{r,S}}\right)\frac{\hat{X}_{t,I}}{B_t}dW^{\mathbb{Q}}_{t,\perp}\\ &= -\phi^I_{t,P}\sigma_r K_{(t,T)}\sqrt{1-\rho^2_{r,S}}\frac{P_{(t,T)}}{B_t}dW^{\mathbb{Q}}_{t,\perp} + \phi^I_{t,I}\rho_{r,I}\sqrt{1-\rho^2_{r,S}}\frac{I_t}{B_t}dW^{\mathbb{Q}}_{t,\perp},\end{aligned} \tag{5.3.4}$$

where the RHS of (5.3.2) implies that $W^{\mathbb{Q}}_{t,\perp}$ only appears in the zero-coupon bond process, and the inflation-indexed bond process. Let us now match the coefficients of the third Brownian motion $W^{\mathbb{Q}}_{t,\top}$, which is only contained in the inflation-indexed bond process. Hence, we obtain:

$$\left(\frac{1}{\gamma}\zeta_\top + \frac{\gamma-1}{\gamma}\sigma_i\sqrt{1-\rho^2_{r,I}}\right)\frac{\hat{X}_{t,I}}{B_t}dW^{\mathbb{Q}}_{t,\top} = \phi^I_{t,I}\sigma_i\sqrt{1-\rho^2_{r,I}}\frac{I_t}{B_t}dW^{\mathbb{Q}}_{t,\top}. \tag{5.3.5}$$

Combine equation (5.3.3) and (5.3.5) into an equivalent matrix equation. This yields,

$$
\begin{bmatrix}
\sigma_s \frac{S_t}{B_t} & -\sigma_r K_{(t,T)} \rho_{r,S} \frac{P_{(t,T)}}{B_t} & \sigma_i \rho_{r,I} \rho_{r,S} \frac{I_t}{B_t} \\
0 & -\sigma_r K_{(t,T)} \sqrt{1-\rho_{r,S}^2} \frac{P_{(t,T)}}{B_t} & \sigma_i \rho_{r,I} \sqrt{1-\rho_{r,S}^2} \frac{I_t}{B_t} \\
0 & 0 & \sigma_i \sqrt{1-\rho_{r,I}^2} \frac{I_t}{B_t}
\end{bmatrix}
\begin{bmatrix}
\phi_{t,S}^I \\ \phi_{t,P}^I \\ \phi_{t,I}^I
\end{bmatrix}
=
\begin{bmatrix}
\left(\frac{1}{\gamma}\zeta_S + \frac{\gamma-1}{\gamma}\sigma_i \rho_{r,I}\rho_{r,S}\right)\frac{\hat{X}_{t,I}}{B_t} \\
\left(\frac{1}{\gamma}\zeta_\perp + \frac{\gamma-1}{\gamma}\sigma_i \rho_{r,I}\sqrt{1-\rho_{r,S}^2}\right)\frac{\hat{X}_{t,I}}{B_t} \\
\left(\frac{1}{\gamma}\zeta_\top + \frac{\gamma-1}{\gamma}\sigma_i \sqrt{1-\rho_{r,I}^2}\right)\frac{\hat{X}_{t,I}}{B_t}
\end{bmatrix}. \tag{5.3.6}
$$

Note that $\zeta_\top$ only depends on the last entry of the above upper triangular system. Assume that the market is complete, and inflation-indexed bonds are traded in the market. Note that the equivalent martingale measure is unique in the complete market, and we derived unique values for market price of risks ζ_S, $\zeta_\perp$, and $\zeta_\top$ in (2.2.6), (2.3.11), and (4.3.13) respectively. If we solve the matrix equation (5.3.6), an explicit solution of the hedging strategy $\phi^I = \{\phi_{t,S}^I, \phi_{u,P}^I, \phi_{t,I}^I\}$ can be found. Additionally, the hedging strategy for the money market account is as follows:

$$\phi_{t,B}^I = \hat{X}_{t,I} - \phi_{t,S}^I S_t - \phi_{u,P}^I P_{(u,T)} - \phi_{t,I}^I I_t. \tag{5.3.7}$$

Let us look at the incomplete market case now. As inflation-indexed bonds are not traded in the incomplete market, the hedging strategy $\phi_{t,I}^I$ for inflation-indexed bonds should identically be equal to 0. Hence, we obtain the worst-case market price of risk $\zeta_\top^*$ as follows:

$$\zeta_\top^* = (1-\gamma)\sigma_i \sqrt{1-\rho_{r,I}^2} < 0. \tag{5.3.8}$$

Notice that we found a negative market price of risk $\zeta_\top^*$ in (5.3.8). As the hedging demand depends positively on the market price of risk, a negative value of $\zeta_\top^*$ indicates that the corresponding asset category, the inflation-indexed bond, is unattractive for investors to hedge. In this case, the expected utility will be pulled down because of the negative market price of risk $\zeta_\top^*$. Let us now find the hedging strategy in the incomplete market. In this case, the hedging strategy does not depend on the third Brownian motion $W_{t,\top}^{\mathbb{Q}}$, and $\phi_{t,I}^I$ is zero. The original upper triangular system (5.3.6) can be reduced into a 2 by 2 matrix system as follows:

$$\begin{bmatrix} \sigma_s \frac{S_t}{B_t} & -\sigma_r K_{(t,T)} \rho_{r,S} \frac{P_{(t,T)}}{B_t} \\ 0 & -\sigma_r K_{(t,T)} \sqrt{1-\rho_{r,S}^2} \frac{P_{(t,T)}}{B_t} \end{bmatrix} \begin{bmatrix} \phi_{t,S}^I \\ \phi_{t,P}^I \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{\gamma} \zeta_S + \frac{\gamma-1}{\gamma} \sigma_i \rho_{r,I} \rho_{r,S} \right) \frac{\hat{X}_{t,I}}{B_t} \\ \left(\frac{1}{\gamma} \zeta_\perp + \frac{\gamma-1}{\gamma} \sigma_i \rho_{r,I} \sqrt{1-\rho_{r,S}^2} \right) \frac{\hat{X}_{t,I}}{B_t} \end{bmatrix}. \tag{5.3.9}$$

Solve the above matrix equation (5.3.9), and we obtain the hedging strategies for stocks and zero-coupon bonds in the incomplete market as follows:

$$\phi_{t,S}^I = \hat{X}_t \frac{1}{S_t} \left[\frac{1}{\gamma} \left(\frac{\lambda}{\sigma_s^2} - \frac{\rho_{r,S}}{1-\rho_{r,S}^2} \left(\frac{(\bar{r}^* - \bar{r})\kappa}{\sigma_r \sigma_s} - \frac{\rho_{r,S}\lambda}{\sigma_s^2} \right) \right) \right], \tag{5.3.10}$$

and,

$$\phi_{t,P}^I = -\hat{X}_t \frac{1}{P_{(t,T)}} \frac{1}{K_{(t,T)}} \left[\frac{1}{\gamma} \frac{1}{1-\rho_{r,S}^2} \left(\frac{(\bar{r}^* - \bar{r})\kappa}{\sigma_r^2} - \frac{\rho_{r,S}\lambda}{\sigma_s \sigma_r} \right) + \frac{\gamma-1}{\gamma} \frac{\sigma_i}{\sigma_r} \rho_{r,I} \right]. \tag{5.3.11}$$

Notice that the hedging strategies for stocks $\phi_{t,S}^C$ (5.1.6) and $\phi_{t,S}^I$ (5.3.10) under the cash and the inflation-indexed bond benchmark are the same. When $\gamma \to \infty$, investors will not invest in the stock market. Furthermore, equation (5.3.11) indicates that the optimal hedging strategy for zero-coupon bonds $\phi_{t,P}^I$ becomes proportionally to parameter σ_r, σ_i, and $\rho_{r,I}$ when investors are extremely risk-averse. To avoid deviating from the inflation-indexed bond benchmark, investors choose the hedging strategy depending on the volatility of the real interest rate, the volatility of inflation-indexed bonds, the correlation parameter.

Chapter 6

Conclusion

This paper has investigated the Martingale method to find optimal wealth and optimal hedging strategies for investors who aim to maximise their utilities under different benchmarks. The basic financial setting is the Black-Scholes-Vasicek economy. The paper has presented the solution of optimal wealth under a general utility function by the Martingale method, and then we have applied it to a power utility function. Furthermore, we have discussed optimal wealth and hedging strategies under different benchmarks in both complete and incomplete markets.

In the paper, three chosen benchmarks are a cash, a stock, and an inflation-indexed bond. The equivalent martingale measure is unique in complete markets. Therefore, the optimal hedging strategies that we have derived under the cash and the stock benchmark are indeed unique and optimal. We have considered the inflation risk in incomplete markets. If inflation-indexed bonds are traded in the market, then investors can obtain a unique pricing kernel to discount asset payoffs. In this case, the market is still complete, and the optimal hedging is unique. However, if inflation-indexed bonds can not be traded, we can determine the optimal hedging under a worst-case martingale measure.

Under the power utility, optimal hedging strategies highly depend on the degree of risk-aversion of investors. Investors with high-degree risk-aversion will increase the portfolio weight of risk-free money market accounts. Optimal hedging strategies are also strongly influenced by the volatility of the benchmark. We have analysed the situation of extreme risk-aversion. In this case, investors are highly concerned about deviating the benchmark. Under the constant benchmark, an extremely risk-averse investor will not invest in stocks and zero-coupon bonds. We have shown that optimal hedging strategies are proportional to the volatility of stocks under the stock benchmark, and to the volatility of stocks the volatilities of the real interest rate and inflation-indexed bonds.

One of the main limitations of the paper is the simplicity of power utility which only includes the degree of risk-aversion of investors. Further research can investigate optimal wealth and hedging under different utility functions and benchmarks.

References

Adam-Müller, A. F. (2000). Hedging price risk when real wealth matters. *Journal of International Money and Finance*, *19*(4), 549–560.

Al-Mohy, A. H., & Higham, N. J. (2009). Computing the fréchet derivative of the matrix exponential, with an application to condition number estimation. *SIAM Journal on Matrix Analysis and Applications*, *30*(4), 1639–1657.

Angelini, F., & Herzel, S. (2014). Delta hedging in discrete time under stochastic interest rate. *Journal of Computational and Applied Mathematics*, *259*, 385–393.

Biagini, F., Guasoni, P., & Pratelli, M. (2000). Mean-variance hedging for stochastic volatility models. *Mathematical Finance*, *10*(2), 109–123.

Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of political economy*, *81*(3), 637–654.

Brennan, M. J., & Xia, Y. (2002). Dynamic asset allocation under inflation. *The journal of finance*, *57*(3), 1201–1238.

Browne, S. (1995). Optimal investment policies for a firm with a random risk process: exponential utility and minimizing the probability of ruin. *Mathematics of operations research*, *20*(4), 937–958.

Chen, Z., Li, Z., Zeng, Y., & Sun, J. (2017). Asset allocation under loss aversion and minimum performance constraint in a dc pension plan with inflation risk. *Insurance: Mathematics and Economics*, *75*, 137–150.

Cox, J. C., & Huang, C.-f. (1989). Optimal consumption and portfolio policies when asset prices follow a diffusion process. *Journal of economic theory*, *49*(1), 33–83.

Dana, R.-A., & Jeanblanc, M. (2007). *Financial markets in continuous time*. Springer Science & Business Media.

Eksi, Z. (2007). A black scholes like model with vasicek interest rates. *Vienna Institute of Finance*.

El Karoui, N., & Quenez, M.-C. (1995). Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM journal on Control and Optimization*, *33*(1), 29–66.

Föllmer, H., & Leukert, P. (1999). Quantile hedging. *Finance and Stochastics*, *3*(3), 251–273.

Föllmer, H., & Leukert, P. (2000). Efficient hedging: cost versus shortfall risk. *Finance and Stochastics*, *4*(2), 117–146.

Gabih, A., Grecksch, W., & Wunderlich, R. (2005). Dynamic portfolio optimization with bounded shortfall risks. *Stochastic analysis and applications*, *23*(3), 579–594.

Girsanov, I. V. (1960). On transforming a certain class of stochastic processes by absolutely continuous substitution of measures. *Theory of Probability & Its Applications*, *5*(3), 285–301.

Goutte, S. (2013). Pricing and hedging in stochastic volatility regime switching models.

Halmos, P. R., Savage, L. J., et al. (1949). Application of the radon-nikodym theorem to the theory of sufficient statistics. *The Annals of Mathematical Statistics*, *20*(2), 225–241.

Harrison, J. M., & Kreps, D. M. (1979). Martingales and arbitrage in multiperiod securities markets. *Journal of Economic theory*, *20*(3), 381–408.

Hull, J., & White, A. (1993). One-factor interest-rate models and the valuation of interest-rate derivative securities. *Journal of financial and quantitative analysis*, *28*(2), 235–254.

Jacka, S. D. (1992). A martingale representation result and an application to incomplete financial markets. *Mathematical Finance*, *2*(4), 239–250.

Jouini, E., & Kallal, H. (1995). Martingales and arbitrage in securities markets with transaction costs. *Journal of Economic Theory*, *66*(1), 178–197.

Kallsen, J., Muhle-Karbe, J., & Vierthauer, R. (2014). Asymptotic power utility-based pricing and hedging. *Mathematics and financial economics*, *8*(1), 1–28.

Karatzas, I. (1997). *Lectures on the mathematics of finance* (Vol. 8). American Mathematical Soc.

Karatzas, I., Lehoczky, J. P., & Shreve, S. E. (1987). Optimal portfolio and consumption decisions for a "small investor" on a finite horizon. *SIAM journal on control and optimization*, *25*(6), 1557–1586.

Mamon, R. S. (2004). Three ways to solve for bond prices in the vasicek model. *Advances in Decision Sciences*, *8*(1), 1–14.

Mania, M., & Tevzadze, R. (2008). Backward stochastic partial differential equations related to utility maximization and hedging. *Journal of mathematical sciences*, *153*(3), 291–380.

Mathias, R. (1992). Evaluating the fréchet derivative of the matrix exponential. *Numerische Mathematik*, *63*(1), 213–226.

Matsumoto, K. (2006). Optimal portfolio of low liquid assets with a log-utility function. *Finance and Stochastics*, *10*(1), 121–145.

Merton, R. C. (1969). Lifetime portfolio selection under uncertainty: The continuous-time case. *The review of Economics and Statistics*, 247–257.

Omberg, E. (1999). Non-myopic asset-allocation with stochastic interest rates. *Working pa*.

Pang, T. (2006). Stochastic portfolio optimization with log utility. *International Journal of Theoretical and Applied Finance*, *9*(06), 869–887.

Pliska, S. R. (1986). A stochastic calculus model of continuous trading: optimal portfolios. *Mathematics of operations research*, *11*(2), 371–382.

Pochart, B., & Bouchaud, J.-P. (2004). Option pricing and hedging with minimum local expected shortfall. *Quantitative Finance*, *4*(5), 607–618.

Primbs, J. A. (2016). *A factor model approach to derivative pricing*. CRC Press.

Privault, N. (2013). *Stochastic finance: an introduction with market examples*. CRC Press.

Renault, E., & Touzi, N. (1996). Option hedging and implied volatilities in a stochastic volatility model 1. *Mathematical Finance*, *6*(3), 279–302.

Sharpe, W. F. (1966). Mutual fund performance. *The Journal of business*, *39*(1), 119–138.

Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of financial economics*, *5*(2), 177–188.

Yang, Z., Ewald, C., & Schenk-Hoppé, K. (2010). An explicit expression to the locally r-minimizing hedge of a european call in the hull and white model. *Quantitative and Qualitative Analysis in Social Sciences*, *4*(1), 1–18.