Network for Studies on Pensions, Aging and Retirement

NETSPAR ACADEMIC SERIES

Welfare loses of a ‘one size fits all’ pension contract for agents with interest rate risk

Part of PhD thesis “Individualized pension contracts: risks, welfare losses and investment choices”

Bart Dees, Theo Nijman, Ole Wilms

DP 05/2024-004

Chapter 3

Welfare loses of a ‘one size fits all’ pension contract for agents with interest rate risk*

*This paper is written together with Theo Nijman and Ole Wilms. We thank Marcus Haveman, Servaas van Bilsen, Frank de Jong, Lieske Coumans and Lennart Dekker for their useful input and feedback.

3.1 Introduction

In the Netherlands pension funds with Defined Benefit (DB) schemes elicit the risk aversion of their participants in order to determine one collective investment strategy. Pension providers with Defined Contribution (DC) schemes likewise elicit the risk aversion of individuals to set a default investment strategy. In these DC pension plans individuals can make their own investment choices, although many individuals end up in the default. With the new Dutch pension law, which is fully effective as of 2028, a much larger group of individuals is expected to have to choose their investment strategy. This individualisation of pension plans highlights the importance of calculating the welfare losses for these individuals of ending up in a 'one size fits all' pension contract. Not only in the Netherlands, but worldwide, we see a shift from collective pension plans towards more individualized pension plans with more investment choices for the participants.
This paper derives the optimal pension contract in a setting with CRRA utility, different assumptions on labor income and interest rate risk for a representative agent taking into account both the accumulation and retirement phase. A key insight is that if this 'one size fits all' pension contract is enforced for agents that deviate substantially from the representative agent in terms of preference parameters and human capital there can be substantial welfare losses. Our setting is different from the literature that models one optimal pension contract for heterogeneous agents. We impose one pension contract on different people. This pension contract is exogenous and we calculate the welfare losses for different people with different characteristics. We consider a range of plausible values for each characteristic, reflecting a low, medium and high risk averse individual. This gives a range for the welfare losses for different specifications of the characteristics and preference parameters of the agent. Our analysis shows that pension providers can realize potentially large welfare gains for their clients by tailoring the pension contract to individual characteristics and preferences.[1]

[1]Note that we assume that if more diverse contracts are offered either the fund observes the key parameters per individual without error or the individual takes rational decisions based on these parameters.

We assume a financial market with two state variables: the short-term interest rate and the stock. The asset menu consists of a stock, a bond and cash. We expand on Van Bilsen et al. (2020) by making human capital more realistic in the model in two ways. Firstly, we determine the optimal life cycle for a representative agent with a career path in line with Cocco et al. (2005). We deviate from Cocco et al. (2005) by assuming constant social benefits in retirement, independent of labour history such that it can be interpreted as a state pension. Secondly, we determine the optimal life cycle for an individual with permanent income shocks and exposure of permanent income shocks to stock market risk. This setting is also inspired by Cocco et al. (2005), who consider a similar setting with constant interest rates.

A novelty of our paper is that we add welfare computations to our setup which previously has only been done for the case without interest rate risk (Bovenberg et al., 2007). We consider three cases under which we analyze welfare losses in the scenario without portfolio constraints (NC). First, we compute welfare losses for the case where a 'one size fits all' pension contract, in terms of consumption and asset allocation, is imposed on an agent who differs with regard to preference parameters (see Section 3.4.1). Second, we quantify the welfare loss of a life cycle strategy that inadequately depends on age only (see Section 3.4.2). Third, we quantify welfare losses of a 'one size fits all' asset allocation based on a standard human capital path, while consumption is optimal (see Section 3.5). In order to address the second case we determine the optimal life cycle for the reference agent which is wealth and age dependent and we denote this by wealth dependent strategy. The industry typically implements a life cycle that depends on age only. We define this age-dependent life cycle as the median of the wealth-dependent life cycle based on wealth simulations. We calculate welfare losses under the restriction that the asset allocation cannot depend on the stochastic development of financial and human wealth.[2] Since pension providers restrict themselves to age-dependent life cycles it is highly relevant to investigate the welfare losses of these life cycles in a setting with interest rate risk.

[2]See for example Bovenberg et al. (2007) for similar calculations in a simplified financial market model.

We consider different sets of portfolio restrictions in calculating the optimal pension contract for three scenarios: no constraints (NC); strictly constraining stock, bond and cash allocations between zero and one (SC); and mild constraints allowing only stock and bond allocation between zero and one and allowing for the shorting of cash (MC). We solve (SC) and (MC) using numerical techniques (see for example Van Bilsen et al. (2020), Koijen et al. (2010) and Carroll (2006)). Assume that a younger participant allocates 100 % of financial wealth to equities, which is typically optimal (Bodie et al, 1992), then the participant cannot hedge the interest rate risk of accumulated financial wealth with the available asset menu (stock, bond and cash) in (SC), since a short position in cash is not allowed although this is potentially welfare improving. In (MC) we assume that the asset allocation for the stock and the bond is between zero and one such that the participant can hedge the interest rate risk of accumulated pension wealth by a short position in cash which is allocated to a long term bond. In this way, the pension industry is allowed to set up an interest rate hedge in individualized products. Therefore, this analysis is of great importance to the pension industry.

In addition we calculate welfare losses of imposing constraints in the pension contract. This analysis is not related to the 'one size fits all' pension contract, but to investment rules of the pension contract set by the authorities. Therefore, we assume optimal consumption and optimal asset allocation under the three different set of restrictions as follows. We calculate the welfare loss of imposing strict constraints in the pension contract where the benchmark setting is without constraints (NC vs SC, see Section 3.6.1). We calculate the welfare gain of imposing mild constraints in the pension contract where the benchmark setting is strict constraints (SC vs MC, see Section 3.6.1).

As indicated before we start the analyses of the welfare loss of 'one size fits all' contracts in the case without constraints. We also calculate welfare losses for agents in a setting with portfolio restrictions (see Section 3.6.2) which is the most relevant case to the pension industry. Here we restrict ourselves to the misspecification of preference parameters due to convergence issues for the other cases to be explained in Section 3.6. We calculate welfare losses of a 'one size fits all' pension contract, in terms of consumption and asset allocation, for different levels of risk aversion. The findings of the paper are as follows.

The *first main finding* of this paper is that, if short positions are allowed, welfare losses of a 'one size fits all' pension contract that implements standard preference parameters are substantial in a setting with interest rate risk. We assume equal steps in one over the level of risk aversion to characterize relevant risk aversion levels. We find that using a pension contract based on a risk aversion that is lower than the risk aversion of the individual, leads to larger welfare losses than a pension contract based on a risk aversion that is higher than the risk aversion of the individual. This corresponds to equity risk that is too high and too low respectively. Bovenberg et al. (2007) find a similar result in a setting without interest rate risk. This asymmetric result should be taken into account by pension funds when determining the collective investment strategy and pension providers by setting their defaults. Welfare losses of a 'one size fits all' pension contract, if short positions are not allowed, remain economically significant as far as the use of inadequate risk aversion levels is concerned. We find that the worst welfare loss is 7.44 % within the framework. The economic intuition is that the equity exposure is too high in this case. For the pension industry this implies that activating their participants to determine their risk profile can avoid significant welfare losses.

The *second main finding* of this paper is that a pension contract with mild constraints, that allows interest rate hedging even if all wealth is allocated to equities, is attractive compared to a setting where short positions are not allowed (SC vs MC). We find a welfare gain of 0.68 % for our calibration.

The *third main finding* of this paper, if short positions are allowed and given the other assumptions, is that welfare losses of an inadequate age dependent strategy, typically used in the pension industry, is significant for the reference pension contract compared to a wealth dependent strategy. This welfare loss is in a similar order of magnitude as the maximum welfare loss of an inadequate level of risk aversion within the framework. This is because pension providers usually do not adjust the investment strategy based on the stochastic development of financial wealth and human capital. Assuming that financial wealth is known, human capital is difficult to estimate since it is determined by the future labour income stream and future interest rates.

The *fourth main finding* of this paper is that, if short positions are allowed, tailoring the asset allocation to inadequate assumptions on human capital (level and risk characteristics) while consumption is optimal, are large. We find that a 'one size fits all' pension contract that overestimates the value of human capital for the asset allocation can lead to welfare losses of more than 20 % given the way we modelled the different career paths. The economic intuition for this large welfare loss is excessive risk taking. In the Netherlands, pension funds now have to tailor their investment strategy to the risk aversion of the group, but the assumption on future income is usually fixed at the current income level. Therefore, tailoring the investment strategy to human capital seems at least equally important as using the right risk preferences. Note though that the emphasis in the sector is on adequate measurement of the risk preference.

The structure of this paper is as follows. Section 3.2 presents an overview of the literature. Section 3.3 solves the life cycle model with interest rate risk. Section 3.4 calculates welfare losses of a pension contract optimized for an individual with different preference parameters and inadequate age dependent lifecycle. Section 3.5 solves the life cycle model for the representative agent with different assumptions on human capital and calculates welfare losses of a pension contract optimized for standard human capital assumptions. Section 3.6 will reconsider the analysis of some previous sections by adding portfolio and consumption constraints to the life cycle model. Section 3.7 provides conclusions.

3.2 Literature overview

Our work builds on the life cycle literature where Merton (1969) derives an optimal constant allocation to the risky asset over the life cycle under a number of simplifying assumptions. Under the assumption of risk free human capital, Bodie et al. (1992) derive that an optimal equity exposure in terms of financial wealth decreases with an increase in age. This is the foundation for individualized pension contracts adopting a life cycle investment strategy. Although many papers, for example Cocco et al. (2015) and Campbell and Viceira (2001), derived optimal strategies under more general assumptions, this conventional wisdom appears to be robust.[3]

There is less consensus in the literature on the optimal interest rate risk the investor faces over the life cycle. Campbell and Viceira (2001) argue that in a setting with time variation in interest rates, an inflation-indexed bond is the risk-free asset for long-term investors; they further show that a nominal bond is an attractive substitute for an inflation-indexed bond if inflation risk is low, though it is a less desirable asset class in an economy with persistent inflation. Brennan and Xia (2002) derive the optimal bond portfolio in a setting with equity risk, real interest rate risk and inflation risk. Brennan and Xia (2002) and Koijen et al. (2010) consider both a two-factor model for interest rates. Brennan and Xia (2002) just like Campbell and Viceira (2002) use the real interest rate and expected inflation as factors, and assume a constant bond risk premium. Koijen et al. (2010) extend Brennan and Xia (2002) by considering a model with time variation in the bond risk premium. Koijen et al. (2010) solve the life cycle optimization with optimal asset allocation to stocks, bonds and cash under borrowing and short sale constraints. Their main result is that as of age 45 the size of the optimal bond portfolio is highly dependent on the bond risk premium at that time. Van Bilsen et al. (2020) try to match the observed duration found in Target Date Funds (TDFs) with a standard life cycle model extended by human capital, inflation and portfolio restrictions. Although they make several extensions to the standard life cycle model, this cannot explain the observed duration in TDFs. The moment they add (risk free) human capital, borrowing and short sale constraints to the model and the agent is able to invest in stocks, a long term bond and cash, the optimal allocation for agents younger than 45 years old is generally stocks and the duration of the optimal financial wealth portfolio is matched with the duration of

[3]Benzoni et al. (2007) is an exception: they find a hump-shaped optimal risk exposure with age, under the assumption of cointegration between labour income and the stock market. Later in this section, we also discuss Branger et al. (2019) where the optimal risk exposure can be increasing in age due to employment risk.

TDFs. In TDFs they also observe a constant duration after age 45, whereas Van Bilsen et al. (2020) show that the optimal duration is upward sloping in the next 20 years and downward sloping afterwards.

Our work is also related to the literature that derives the optimal asset allocation for a representative agent under a variety of assumptions on human capital. Cocco et al. (2005) determine the optimal asset allocation for a representative agent with human capital, but the assumptions on human capital differ, in the absence of interest rate risk. Their main contribution is that labour income, although it is risky, increases the allocation to stocks compared to the setting without labour income (Merton, 1969). Cocco et al. (2005) model human capital by adding a career path, an idiosyncratic temporary income shock and a permanent shock consisting of an idiosyncratic part and an aggregate component that is correlated with the financial market. Participants with steeper career paths retain a higher equity exposure in terms of financial wealth during midlife than participants with less steeper career paths due to a higher exposure to the risk free asset via a claim on future labour income. In Cocco et al. (2005), a steeper career path can lead to 10-15 % higher optimal equity exposure. A higher variance of the temporary and permanent shock leads to higher labour income risk which increases the precautionary savings motive and crowds out portfolio risk. In Cocco et al. (2005) this can be 5-25 % lower equity exposure around midlife. Labour income correlated with the stock market lowers the benefits of investing in equities. Cocco et al. (2005) show that this is 20 % less equity exposure at the start of the career for a correlation coefficient of 0.2. Another effect is that saving becomes less attractive leading to lower wealth accumulation and hence a higher equity exposure in terms of financial wealth from midlife onwards. Their extension of the model with a probability of a disastrous labour income shock reduces the allocation to stocks significantly. They allow for a zero labour income with 0.5 % probability leading to an average stock exposure of 45 % at age 30, whereas the benchmark individual has 100 % equity exposure. The stock exposure in terms of financial wealth of older investors is affected to a lesser degree due to higher financial wealth accumulation (precautionary savings motive).

Munk and Sørensen (2010) solve the optimal asset allocation problem in a setting with stochastic interest rates and stochastic labour income. They model labour income with a geometric brownian motion with the drift term an affine function in the interest rates allowing for business cycle sensitivity. Their model also takes income stock and income bond correlations into account. They find that the optimal allocation to stocks, bonds and cash is significantly affected by labour income uncertainty and the movement of labour income with business cycle variations. This slope coefficient, which differs across participants, determines the extent to which human capital reacts to business cycle variations and therefore whether human capital is a close substitute for cash or a long-term bond. This has a direct impact on the optimal demands for stocks and especially cash and bonds. If the slope coefficient is close to zero, human wealth is non-cyclical and a close substitute for a long-term bond, therefore increasing the optimal demand for cash and decreasing the optimal demand for long-term bonds. If the slope coefficient is close to one, human wealth is cyclical and a close substitute for cash therefore increasing the optimal demand for long-term bonds and decreasing the optimal demand for cash. Branger et al. (2019) show that adding unemployment risk to the model reduces the optimal demand for stocks in a setting without interest rate risk. The risk of getting unemployed varies with age and with the business cycle. Especially for young individuals the probability of getting unemployed makes human capital risky since it can lead to lower future labour income. Human capital is also correlated with the financial market since un- and re-employment probabilities are correlated with the state of the economy. This leads to the conclusion that in a model with unemployment risk the optimal allocation to stocks is no longer decreasing in age. In fact, the optimal allocation to stocks is upward sloping over the life cycle and has extremely low allocations to risky asset at young ages compared to what the literature, for example Cocco et al. (2005), typically finds. Other than the variable age, there is no heterogeneity across individuals in the probability of getting unemployed.

3.3 Solving a life cycle model with interest rate risk

In order to be able to compute welfare losses of suboptimal pension contracts we first of all have to determine what is optimal for the specific characteristics at hand. We describe the financial market in Section 3.3.1. Section 3.3.2 derives the optimal asset allocation in terms of total wealth. Section 3.3.3 analyzes the optimal pension contract (contribution rates, life cycle strategy for equity allocation, as well as interest rate hedge per age group) assuming risk-free constant human capital. We impose no borrowing constraints. The model in Section 3.3 is identical to Van Bilsen et al. (2020). Note that the welfare computations have been added. For Section 3.4 and 3.5 we extend the model of Van Bilsen et al. (2020). Section 3.4 analyzes welfare losses of a ‘one size fits all pension’ contract for inadequate preference parameters and an inadequate age dependent life cycle. Section 3.5 analyzes welfare losses of an inadequate age dependent life cycle.

3.3.1 Financial market

We assume a financial market with two state variables: the short-term interest rate and the stock. We assume that the asset menu consists of a stock, a bond and cash. We choose to disregard inflation, and assume that inflation risk is negligible and such that a nominal long term bond is the risk free asset (Campbell and Viceira, 2001). Just like many papers in the literature we assume that the individual has constant relative risk aversion (CRRA) preferences over consumption $c(t)$

$$U = E\left[\int_0^T \exp\{-\delta t\}\frac{1}{1-\gamma}c(t)^{1-\gamma}dt\right] \tag{3.1}$$

with γ being the risk preference parameter of the individual and δ being the subjective time preference of the individual. T is the sum of the years in the working phase and retirement phase. We define the process for the real instantaneous interest rate and stock return as follows. We have a one-factor model for the real interest rate $r(t)$

$$dr(t) = \kappa_r(\overline{r} - r(t))dt + \sigma_r dZ_r(t) \tag{3.2}$$

$$dS(t) = \Big(r(t) - \lambda_r\sigma_r D_s(t) + \lambda_s\sigma_s\Big)S(t)dt + \sigma_s S(t)dZ_s(t) - D_s(t)\sigma_r dZ_r(t) \tag{3.3}$$

where we define $\overline{r}$ as the expected short-term interest rate at the long term and κ_r as the mean reversion coefficient. The volatility of interest rates and stocks is defined by σ_r and σ_s respectively. The duration of stocks is denoted by $D_s(t)$. The market prices of stock market risk is defined by λ_s. We define $Z_s(t)$ and $Z_r(t)$ as independent brownian motions. Note that equations (3.2) and (3.3) differ from Campbell and Viceira (2001) and Brennan and Xia (2002) who use a two-factor model for the nominal rate (namely the sum of real rate and expected inflation). We define the stochastic discount factor $M(t)$

$$\frac{dM(t)}{M(t)} = -r(t)dt + \phi' dZ(t) \tag{3.4}$$

with $Z(t) = (Z_r(t), Z_s(t))$ and $\phi = (\phi_r, \phi_s)$. We define the market prices of risk, λ_r and λ_s, as follows

$$\lambda_r = -\phi_r \tag{3.5}$$

$$\lambda_s = -\phi_s. \tag{3.6}$$

The price of a bond $p(t, h)$ with maturity h at time t evolves over time

$$\frac{dp(t,h)}{p(t,h)} = \Big(r(t) - \lambda_r\sigma_r B_r(h)\Big)dt - B_r(h)\sigma_r dZ_r(t) \tag{3.7}$$

with $B_r(h) = \frac{1-\exp\{-\kappa_r h\}}{\kappa_r} \in [0, h]$ the interest rate duration of the bond. We define these bonds in real terms. The dynamics of wealth $W(t)$ evolve with portfolio weights $\omega(t)$

$$dW(t) = \Big(r(t) + \omega(t)'\{\mu(t) - r(t)\}\Big)W(t)dt + \omega(t)'\Sigma(t)W(t)dZ(t) - c(t)dt \tag{3.8}$$

and $\mu(t)$ and $\Sigma(t)$ as follows

$$\mu(t) = \begin{pmatrix} r(t) - \lambda_r\sigma_r B_r(h) \\ r(t) - \lambda_r\sigma_r D_s(t) + \lambda_s\sigma_s \end{pmatrix} \tag{3.9}$$

$$\Sigma(t) = \begin{pmatrix} -B_r(h)\sigma_r & 0 \\ -D_s(t)\sigma_r & \sigma_s \end{pmatrix}. \tag{3.10}$$

We define the parameters for the financial market and individual characteristics in Table 3.1 almost identical to the benchmark specification as in Van Bilsen et al. (2020). The only difference is that we set the interest rate duration of stocks $D_s(t)$ equal to zero in line with the standard in the literature. We discretize the model with time steps dt of 1/100 year and consider individuals over their complete life span. Although there is a closed form solution in this unconstrained setting we discretize for the following reasons. We want to keep results comparable with later Sections for which we need discretizations as well as numerical approximations. Note also that in practice decision making obviously occurs in discrete time. Discretization in the unconstrained case is also in line with Bovenberg et al. (2007).

[4] As a clarification we cite footnote 10 of Van Bilsen et al. (2020): 'The half-time of the interest rate η is the time it takes for the interest rate to revert half way back to its long-term mean from its current level if no Brownian shocks arrive. The mean reversion coefficient κ_r can be computed from the half-time of the interest rate as follows: $\kappa_r = \frac{\log(2)}{\eta}$.'

Name	Parameter	Value
Starting age	T_s	20
Working years	T_r	45
Working and retirement years	T	65
Risk aversion parameter	γ	5
Time preference parameter	δ	0.03
Expected long-run short term interest rate	$\bar{r}$	0.02
Interest rate volatility	σ_r	0.01
Mean reversion coefficient	κ_r	$\frac{\log(2)}{20}4$
Interest rate risk premium	λ_r	- 0.075
Interest rate duration of stocks	$D_s(t)$	0
Stock price volatility	σ_s	0.18
Market price of equity risk	λ_s	0.2

Table 3.1: Overview of parameter values.

3.3.2 Optimal asset allocation of total wealth

The maximization problem of the individual, in which we impose no constraints (NC), is as follows

$$\max_{c(t),\omega(t)} E\left[\int_0^T \exp\{-\delta t\}\frac{1}{1-\gamma}c(t)^{1-\gamma}dt\right] \tag{3.11}$$

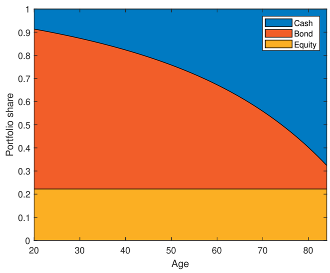
$$\text{s.t. } dW(t) = \Big(r(t) + \omega(t)'\{\mu(t) - r(t)\}\Big)W(t)dt + \omega(t)'\Sigma(t)W(t)dZ(t) - c(t)dt. \tag{3.12}$$

The optimal portfolio weights to stocks and bonds in terms of total wealth are given by $\omega_s^*(t)$ and $\omega_p^*(t)$. There is a speculative demand for stocks and for bonds there is a speculative demand as well as a hedging demand. Derivations can be found in **Appendix A1**

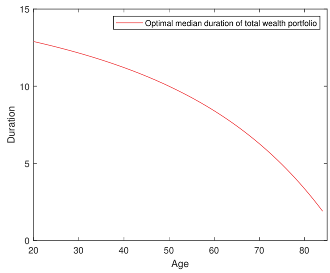
$$\omega_s^*(t) = \underbrace{-\frac{1}{\gamma}\frac{\phi_s}{\sigma_s}}_{\text{speculative demand}} \tag{3.13}$$

$$\omega_p^*(t) = \underbrace{\frac{1}{\gamma}\frac{\phi_r}{B_r(h)\sigma_r}}_{\text{speculative demand}} + \underbrace{\frac{D_A(t)}{B_r(h)}}_{\text{hedging demand}} . \tag{3.14}$$

We present the optimal asset allocation in Figure 3.1a and the optimal duration of the wealth portfolio in Figure 3.1b. This is a replication of what Van Bilsen et al. (2020) find. We find in Figure 3.1b a decreasing duration for the total wealth portfolio due to less bond and more cash exposure over time in Figure 3.1a.



(a) Optimal asset allocation in terms of total wealth



(b) Optimal duration of total wealth portfolio

Figure 3.1: The figure shows the optimal asset allocation and optimal duration of the total wealth portfolio in a median scenario for the model of Van Bilsen et al. (2020). For example, this implies that we assume $\gamma = 5$ and a bond with maturity $h = 30$. The full parameterization is presented in Table 3.1.

We present optimal consumption $c^*(t)$ as a function of total wealth $W(t)$ and $A^*(t)$ the wealth consumption ratio at time t with details and the derivation in **Appendix A1**

$$c^*(t) = \frac{W(t)}{A^*(t)}. \tag{3.15}$$

We show the optimal consumption level in nominal terms in Figure 3.2. Although the time preference parameter is higher than the median interest rate, optimal median consumption increases over time due to the risk premium in the risky asset. If we assume that inflation is a bit less than 1 % this yields real flat median consumption.

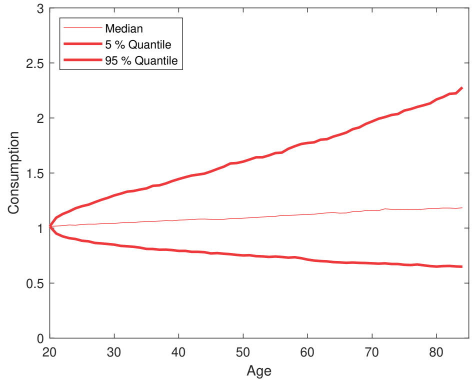


Figure 3.2: The figure shows the consumption distribution for the model of Van Bilsen et al. (2020). The full parameterization is presented in Table 3.1.

3.3.3 Optimal asset allocation of financial wealth

We normalize labour income before retirement age to 1 and after the fixed retirement age T_r the retiree receives a state pension s. This is defined in the variable $O(t+h)$

$$O(t+h) = \begin{cases} 1 & \text{if } t+h < T_r \\ \text{s} & \text{if } t+h \geq T_r. \end{cases} \tag{3.16}$$

We define the value of the contribution to human capital h periods from time t by $L(t,h)$ as follows

$$L(t,h) = E_t\left[\frac{M(t+h)}{M(t)} O(t+h)\right] \tag{3.17}$$

and taking the integral over h defines $L(t)$, the value of human capital at time t

$$L(t) = \int_0^{T-t} L(t,h)dh. \tag{3.18}$$

The optimal portfolio weights under a stochastic saving decision is given by $\hat{\omega}_s(t)$ and $\hat{\omega}_p(t)$ in terms of financial wealth. The second term in (3.20) is a correction term for the duration that human capital already adds to the portfolio. The derivations can be found in **Appendix A2**

$$\hat{\omega}_s(t) = \frac{W(t)}{F(t)}\omega_s^*(t) \tag{3.19}$$

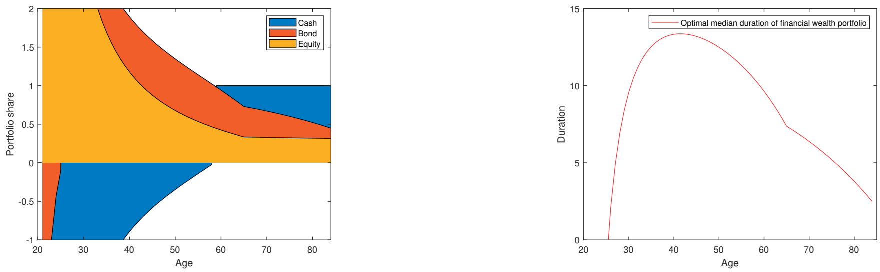
$$\hat{\omega}_p(t) = \frac{W(t)}{F(t)}\omega_p^*(t) - \frac{L(t)}{F(t)}\frac{D_L(t)}{B_r(h)}. \tag{3.20}$$

We present the optimal asset allocation in Figure 3.3*a* and the optimal duration of the financial wealth portfolio in Figure 3.3*b*. This is a replication of what Van Bilsen et al. (2020) find, where the equity exposure decreasing in retirement is explained by the inclusion of the state pension s of 0.4 in the value of human capital. Although the state pension is economically very different from labour income, it plays the role of a fixed income stream in the definition of human capital.

The optimal duration in terms of financial wealth is low or even negative at young ages. In other words, we see a short position in the bond for these cohorts. This is explained by a high value of human capital that already implies more duration than needed to the portfolio in terms of total wealth.[5]

Optimal consumption is in line with equation (3.15). By adding risk free constant human capital, we can define the optimal saving decision at time t as $O(t) - c^*(t)$. We see from optimal consumption in Figure 3.2 that in a median scenario no savings are made. We can explain this by a high return on a long position in stocks financed by a large short position in cash. This investment strategy is

[5]Van Bilsen et al. (2020) also do a sensitivity analysis for the duration of the optimal financial wealth portfolio with respect to the level of the state pension and risk aversion in their Figure 10. If they lower the value of the state pension s to 0.2, which means a lower value of human capital, we see a positive duration of the optimal financial wealth portfolio for the youngest.



(a) Optimal asset allocation of financial wealth with constant risk free human capital with human capital.

(b) Optimal duration of financial wealth portfolio.

Figure 3.3: The figure shows the optimal asset allocation and optimal duration of the financial wealth portfolio in a median scenario for the model of Van Bilsen et al. (2020). We assume $\gamma = 5$ and a bond with maturity $h = 30$. The full parameterization is presented in Table 3.1. We have assumed the dynamics of human capital in line with equation (3.16).

sufficient to finance the consumption standard after retirement age. This situation only happens if short positions are allowed.

We calculate welfare losses from a mismatch in the characteristics of the individual using the method of certainty equivalents. We define the certainty equivalent of consumption ce as the certain consumption stream that makes the individual indifferent between the stochastic consumption stream and the certainty equivalent consumption stream

$$E\left[\int_0^T \exp\{-\delta s\}\frac{1}{1-\gamma}c^*(t)^{1-\gamma}ds\right] = \int_0^T \exp\{-\delta s\}\frac{1}{1-\gamma}ce^{1-\gamma}ds. \tag{3.21}$$

We calculate the optimal certainty equivalent ce^* in continuous time as follows

$$ce^* = \left(\frac{E\left[\int_0^T \exp\{-\delta s\}\frac{1}{1-\gamma}c^*(t)^{1-\gamma}ds\right](1-\gamma)}{\int_0^T \exp\{-\delta s\}ds}\right)^{\frac{1}{1-\gamma}}. \tag{3.22}$$

This enables us to calculate welfare losses $loss$ as follows,

$$loss = \frac{ce - ce^*}{ce^*} \tag{3.23}$$

where ce is the certainty equivalent of a suboptimal consumption stream and ce^* is the certainty equivalent of the optimal consumption stream.

3.4 One size fits all contract with standard human capital without constraints

Section 3.4.1 calculates welfare losses of a 'one size fits all' pension contract optimized for the reference agent while individuals differ in 'preference parameters (risk aversion and time preference) no constraints'. Section 3.4.2 calculates the welfare losses of an 'inadequate age dependent lifecycle no constraints'.

3.4.1 Welfare losses of inadequate preference parameters

We characterize agents of six types: a low, medium and high level of risk which we combine with a low and high time preference parameter. We choose levels of risk aversion such that we have almost linear steps in the optimal equity exposure (see Panel A). In Table 3.2 we calculate the welfare loss of a mismatch in the level of risk aversion and time preference parameter due to a 'one size fits all' pension contract.

Evidently the stochastic consumption stream with $\hat{\gamma} = 5$ and $\hat{\delta} = 0.03$ imposed in the pension contract yields a zero welfare loss because the true preference parameters $\gamma = 5$ and $\delta = 0.03$ are used. For example, the welfare loss of a stochastic consumption stream with $\hat{\gamma} = 5$ and $\hat{\delta} = 0.03$ imposed in the pension contract while true preference parameters are $\gamma = 10$ and $\delta = 0.02$ is equal to 9.64 %. The economic reason for this loss is excessive risk taking (too high speculative stock demand). The inadequate investment strategy assumes 22.22 % equity exposure while it is optimal to have 11.11 % equity exposure. In addition, the speculative demand for bonds is also too high with a factor two. The hedging demand term is not affected by γ, but is slightly higher than optimal via δ in $D_L(t)$. Furthermore, the suboptimal time preference parameter leads to a too high consumption standard (or undersaving) in the younger years. We conclude this from the fact

	Panel A: Welfare losses	
	$\delta = 0.02$	$\delta = 0.03$
$\gamma = 3$	-3.24 %	-3.32 %
$\gamma = 5$	-0.33 %	0 %
$\gamma = 10$	-9.64 %	-7.96 %
	Panel B: Optimal equity exposure in terms of total wealth	
	$\omega_s^*(t)$	
$\gamma = 3$	37.04 %	
$\gamma = 5$	22.22 %	
$\gamma = 10$	11.11 %	
	Panel C: Optimal wealth-consumption ratio	
	$\delta = 0.02$	$\delta = 0.03$
$\gamma = 3$	32.30	29.82
$\gamma = 5$	33.57	31.95
$\gamma = 10$	35.10	34.21

Table 3.2: Welfare losses from a pension contract which is optimized for an individual with different risk aversion γ and time preference δ used for both the investment and consumption strategy for a participant with a flat career path (Panel A). We have also included the optimal equity exposure in terms of total wealth (Panel B) and optimal wealth-consumption ratio for different values of preference parameters that we use for inadequate investment and consumption strategies in the first year (Panel C). The financial market parameters are described in Table 3.1. The imposed parameters for the 'one size fits all' pension contract are $\gamma = 5$ and $\delta = 0.03$.

that the suboptimal wealth-consumption ratio is lower than the optimal wealth to consumption ratio (31.95 versus 35.10). The welfare loss of a stochastic consumption stream with $\hat{\gamma} = 5$ and $\hat{\delta} = 0.03$ imposed in the pension contract while true preference parameters are $\gamma = 3$ and $\delta = 0.03$ is equal to 3.32 %. The economic reason for this loss is insufficient risk taking. The inadequate investment strategy assumes 22.22 % equity exposure while it is optimal to have 37.04 % equity exposure (too low speculative stock demand). In addition, the speculative for bonds is also too low with almost a factor two. The hedging demand term is not affected by γ. Even though a suboptimal risk aversion parameter on its own would already have an effect on the consumption strategy, we have additionally chosen an optimal time preference parameter. We observe oversaving in the younger years since the suboptimal wealth consumption ratio is higher than the optimal wealth to consumption ratio (31.95 versus 29.82). The welfare loss of a stochastic consumption stream with $\hat{\gamma} = 5$ and $\hat{\delta} = 0.03$ imposed in the pension contract, while true preference parameters are $\gamma = 5$ and $\delta = 0.02$, is equal to 0.33 %. In this situation the participant has the optimal equity exposure in terms of total wealth, so the equity risk is neither too low nor too high. However, hedging demand term for the bond is slightly higher than optimal via δ in $D_L(t)$. The suboptimal time preference parameter leads to a consumption standard that is too high (or undersaving) in the younger years that we see from the fact that the wealth-consumption ratio is too low. Later on in life this participant reaches a consumption standard lower than optimal. We graphically present this for both contracts (i.e. $\hat{\gamma} = 5$, $\hat{\delta} = 0.03$ and $\hat{\gamma} = 5$, $\hat{\delta} = 0.02$) in Figure 3.4. We conclude that welfare losses of a 'one size

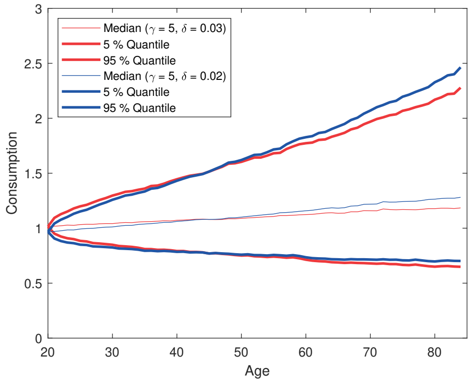


Figure 3.4: The figure shows the consumption distribution for ($\hat{\gamma} = 5$, $\hat{\delta} = 0.03$) and ($\hat{\gamma} = 5$, $\hat{\delta} = 0.02$) for the model of Van Bilsen et al. (2020) where we use the true preference parameters. The full parameterization is presented in Table 3.1.

fits all' pension contract as far as the use of inadequate risk aversion levels leads to higher welfare losses than the use of inadequate preference parameters. Therefore, it is natural that the Dutch pension industry elicit the risk aversion parameter on a frequent basis and prioritize this over the time preference parameter.

We found welfare losses in the same order of magnitude as Bovenberg et al. (2007) who considered the case without interest rate risk. For the same range of preference parameters their paper found a maximum welfare loss of 6.8 %. Joseph, Pelsser and Werner (2021) also calculate losses from suboptimal risk aversion and show that the maximum annual investment return loss per year —which is similar to a welfare loss— is around 2 % for levels of risk aversion within the interval with lower bound 1.5 and upper bound 4. Their paper does not state the parameterization of the financial market model.

3.4.2 Welfare losses of inadequate age dependent life cycle

We have determined the optimal life cycle in a setting with interest rate risk and (just like for the case without interest rate risk) this depends on wealth and age (henceforth wealth dependent life cycle). In this section we consider an age dependent life cycle —typically used by the industry, though suboptimal in theory— as the median of the wealth dependent life cycle. When imposing the restriction that the asset allocation cannot depend on the stochastic development of financial and human wealth while consumption is optimal, we see that this welfare loss is estimated to be 7.65 % for the reference agent. On the one hand, we have that an increase (decrease) of the interest rate decreases (increases) the value of human capital which decreases (increases) the equity exposure in the wealth dependent strategy. On the other hand, we have that an increase (decrease) of the interest rate increases (decreases) the value of financial wealth which decreases (increases) the equity exposure in the wealth dependent strategy. Furthermore, high (low) equity returns increase (decrease) financial wealth which decrease (increase) the equity exposure in the wealth dependent strategy. Therefore, the economic intuition of this welfare loss is explained by the fact that the participant cannot retain the optimal asset allocation in terms of total wealth since he cannot adjust the asset allocation based on the evolution of financial and human wealth.

3.5 One size fits all contract with different assumptions on human capital without constraints

We will extend the model in Section 3.3 by making human capital more realistic step by step. This enables us to obtain a more realistic expression for the optimal asset allocation. In Section 3.5.1 we assume that the agent has a deterministic risk free career path that varies across educational attainment instead of a constant risk free life time labour income. We consider this extension since participants differ in their career path (Cocco et al., 2005), leading to different optimal asset allocations. We characterize agents of three types: a career path based on 'no high school', 'high school' and 'college' education respectively. In Section 3.5.2 we add a permanent shock to human capital consisting of an idiosyncratic component and an aggregate component with correlation to the stock market. An economic example of an idiosyncratic permanent shock to labour income is becoming a (partially) disabled worker. The economic intuition for exposure to a permanent shock correlated with the financial market is that different sectors have different exposures to shocks in the economy. An example with low exposure to stock market risk is a job in public administration, whereas high exposure to stock market risk is a job in the financial sector. We characterize agents of three types: a low, medium and high exposure coefficient. These extensions are relevant such that we can tailor the optimal asset allocation to characteristics of the individual and quantify welfare losses of a 'one size fits all' pension contract based on inadequate assumptions for human capital. We have no restrictions (NC).

A related paper is by Vestman et al. (2021), who optimize the contribution rate yearly on characteristics of participants including age, income and stock market participation using the 100 minus age for the asset allocation. They conclude that this flexible design of contribution rates leads to a welfare gain of 3 % on average. Our paper focuses on the optimal asset allocation, instead of a 100 minus age rule,[6] and assumes the optimal contribution rate for the three type of agents who differ

[6] The '100 minus age rule' means that a percentage of 100 minus the age of the agent is invested in stocks.

in their human capital with interest rate risk included.

3.5.1 Welfare losses of inadequate career path

We allow for a deterministic risk-free career path $f_i(t+h)$ during the accumulation phase, which can be different across individuals i as in Cocco et al. (2005). For convenience we define the labour income with career path $\tilde{O}_i(t+h)$ based on the labour income without career path $O(t+h)$ corresponding to the time index $t=0$ at the start of the career

$$\tilde{O}_i(t+h) = O(t+h) \cdot \underbrace{f_i(t+h)}_{\text{career path}}. \tag{3.24}$$

We can determine $f_i(t+h)$ in line with Cocco et al. (2005), using a third-order polynomial, and we do not reestimate but assume the 2003 pattern for the UK to be relevant for our setting. Details are available in **Appendix B** and the starting age T_s is set to 20

$$f_i(t+h) = \exp\left\{a_{0,i} + a_{1,i} \cdot \left(T_s + h\right) + a_{2,i} \cdot \left(\frac{(T_s+h)^2}{10}\right) + a_{3,i} \cdot \left(\frac{(T_s+h)^3}{100}\right)\right\}. \tag{3.25}$$

We normalize the labour income at the start age with the career path to one by dividing $f_i(t+h)$ in equation (3.25) by $f_i(0)$. We deviate from Cocco et al. (2005) in the sense that for us, 'labour income in retirement' is a constant, independent of past labour career, such that it has the interpretation of a state pension. Cocco et al. (2005) model retirement as a fraction from labour income one year before retirement. We present the different career paths in Figure 3.5.

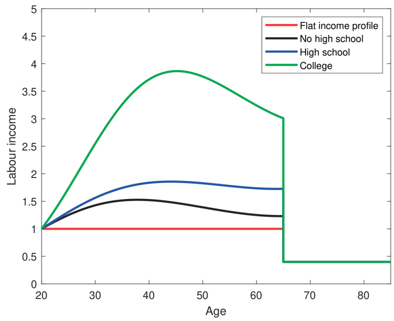


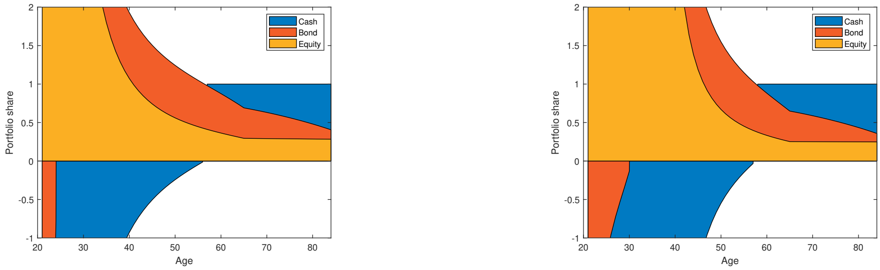
Figure 3.5: We present the career paths for an agent with education levels 'no high school', 'high school' and 'college' based on Cocco et al. (2005). We calculate the career path using equation (3.25) and coefficients $(a_0, a_1, a_2, a_3) = (-2.1361, 0.1684, -0.0353, 0.0023)$, $(a_0, a_1, a_2, a_3) = (-2.1700, 0.1682, -0.0323, 0.0020)$, $(a_0, a_1, a_2, a_3) = (-4.3148, 0.3194, -0.0577, 0.0033)$ for an agent with 'no high school', 'high school' and 'college' respectively. During retirement we assume that the agent receives a risk free social benefit of s=0.4.

We can still calculate the contribution of income h periods from time t to human capital as follows, since $f_i(t+h)$ is a deterministic function

$$L(t,h) = \tilde{O}_i(t+h) \cdot E_t\left[\frac{M(t+h)}{M(t)}\right]. \tag{3.26}$$

Although we add a risk-free career path to human capital—equation (3.26)—via $\tilde{O}_i(t+h)$, the formulas determining the optimal asset allocation are still equations (3.19) and (3.20), but the value of human capital changes from equation (3.17) with $O(t+h)$ to equation (3.26) with $\tilde{O}_i(t+h)$. We present the optimal asset allocation in Figure 3.6 for an agent with educational attainment 'no high school' and 'college' respectively while consumption is optimal.[7] We switch the sign of the exposures if financial wealth is negative such that the reader can interpret the exposures more intuitively. In a median scenario financial wealth becomes non-negative for a participant with 'no high school', 'high school' and 'college' educational attainment at the ages of 20, 28 and 37 respectively. Under the assumption of one of the three

[7] We do not present the optimal asset allocation for an agent with 'high school' education, but we do discuss the results in the main text.



(a) Optimal asset allocation 'no high school' education.

(b) Optimal asset allocation 'college' education.

Figure 3.6: The figure shows the optimal asset allocation in a median scenario while consumption is optimal for the model of Van Bilsen et al. (2020), extended by the career paths from Cocco et al. (2005). We only present the optimal asset allocation for the career paths at the extreme within our framework: 'no high school' and 'college'. The full parameterization is presented in Table 3.1.

career paths defined in Figure 3.5, the claim to a risk-free asset —future labour income— is higher than in the setting of Van Bilsen et al. (2020), where labour income is assumed to be constant during the accumulation phase. We expect that this leads to higher stock exposures at younger ages according to equation (3.19), but this cannot be interpreted straightforwardly from Figure 3.6. The explanation is that adding a risk free career path leads to a higher value of human capital at young ages which leads to a higher consumption standard at young ages. Figure 3.7 shows that for a steeper career path the optimal median consumption is higher.[8] This higher consumption standard implies that accumulated financial wealth is negative in more than half of the scenarios for 'high school' and 'college' career path since the higher consumption standard is financed by borrowing. This negative financial wealth explains the negative sign as a percentage of financial wealth in front of the allocation to risky assets at young ages for a 'high school' and 'college' career path. We show that in the first period by equation (3.19) —when financial wealth is zero— a participant with 'no high school', 'high school' and 'college' have respectively 9.55, 11.27 and 19.68 invested in the stock market financed by

[8]Consumption with a flat income profile was also presented in Figure 3.2. Also in this case almost all labour income is consumed, and therefore savings are small, due to extreme asset allocations with high investment returns.

borrowing. Therefore, the allocation to stocks in euro values is positive for all career paths considered. Furthermore, the steeper the career path, the higher the allocation to stocks in euro values. The economic intuition of negative financial wealth at young ages for higher educational attainment might be economically explained by student loan debt. However, the model allows for negative financial wealth in a higher order of magnitude than observed in the real world.

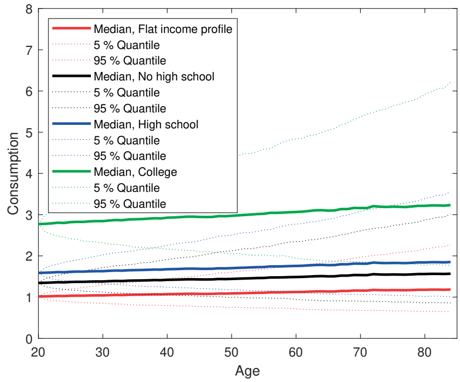


Figure 3.7: The figure shows the consumption distribution for a participant with a career path corresponding to 'no high school', 'high school' and 'college' educational attainment respectively. The full parameterization is presented in Table 3.1.

In Table 3.3 we present the welfare loss of a 'one size fits all pension contract' with a mismatch in the participants risk-free career path which is used for the asset allocation while consumption is optimal. The welfare losses of a mismatch in human capital for the asset allocation while consumption is optimal can be up to 3.69 % for underestimating the value of human capital. For example, when the participant has 'college' educational attainment, but the 'one size fits all' pension contract assumes a life cycle with 'no high school' human capital the welfare loss is highest at 3.69 %. Economically this is explained by a lack of risk taking.
The welfare loss of a mismatch in human capital for the asset allocation while consumption is optimal for overestimating the value of human capital is larger. For example, when the participant has 'high school' educational attainment, but

the 'one size fits all' pension contract assumes a life cycle with 'college' human capital, the welfare loss is highest at 21.86 %. Economically this is explained by excess risk taking. All the welfare losses in this subsection are explained by an inadequate assumption on the ratio of human wealth over financial wealth. Here we assume that the agent has the risk aversion that is used to derive the 'one size fits all' pension contract. So, the only source of the welfare losses is the inadequate ratio of human wealth over financial wealth.

	'No high school'	'High school'	'College'
'No high school'	0 %	-0.42 %	-3.69 %
'High school'	-0.65 %	0 %	-2.45 %
'College'	-54.19 %	-21.86 %	0 %

Table 3.3: Welfare losses for a participant with education level defined in the columns, but the education level in the rows is used as inadequate investment strategy in the 'one size fits all' pension contract while consumption is optimal. The financial market and individual parameters are described in Table 3.1. Figure 3.5 presents the career paths that we assume for the three three agents.

Because we simulate in discrete time total wealth and consumption can get negative in extreme scenario's. This happens for a participant with 'no high school' educational attainment if the 'one size fits all' pension contract assumes a life cycle with 'college' human capital (i.e. increasing income profile). This also happens when we use a daily grid, while continuous time rebalancing should prevent this. This implication of simulating in discrete time is addressed by Branger, Breuer and Schlag (2010). They show that adding a tiny position in far out of the money puts solves the issue and keeps wealth positive. Their result suggests that a similar way to implement this is to drop scenario's where wealth gets below a low cut-off point. The result by Branger et al. (2010) suggests that the exact cut off point hardly impacts the estimate of the welfare loss if the number of simulations is very large. We draw 2,000 scenarios and set the cut-off point at consumption of 0.001 i.e., 0.1% of annual income for an agent with a flat income profile. This implies for the very specific and extreme case discussed here (very inadequate assumption on human capital) that we exclude five scenarios (0.25% of the cases) in which

negative consumption occurs.
The welfare effects we find are larger than Cocco et al. (2005) find for agents who differ in their career path, where the suboptimal ‘100-age’ rule is used for the stock allocation. They find welfare losses in the order of magnitude of 1 %. We find higher numbers since the suboptimal life cycles are extremer in excess risk and too low risk mainly due to the set up and that we abstain from portfolio and consumption restrictions.

3.5.2 Welfare losses of permanent income shocks with inadequate equity exposure

We allow for a permanent shock consisting of an idiosyncratic component ω_{it} with normal distribution $N(0, \sigma_\omega^2)$ and an aggregate component correlated with the financial market based on Cocco et al. (2005). We define $Z_\omega(t)$ as a brownian motion independent of $Z_r(t)$ and $Z_s(t)$. We define $\beta_{s,i}$ as the exposure coefficient of labour income to the stock market for individual i. The economic intuition for a permanent shock with idiosyncratic component is, for example, becoming a (partially) disabled worker. The economic intuition for exposure to a permanent shock correlated with the financial market is that different sectors have different exposures to shocks in the economy. For example, the stock market exposure is lower for an individual working in the sector public administration compared to the finance sector. We denote labour income —in the accumulation phase— in this setting as follows

$$\log(\overline{O}_i(t+h)) = \log(\tilde{O}_i(t+h)) + \underbrace{\sigma_\omega \int_0^{t+h} dZ_\omega(s)}_{\text{perm. shock (idiosyncratic)}} + \underbrace{\beta_{s,i}\sigma_s \int_0^{t+h} dZ_s(s)}_{\text{perm. shock correlated with stocks}} . \tag{3.27}$$

Obviously the value of human wealth changes with a different exposure coefficient. We consider an incomplete market since the participant cannot hedge shocks in $Z_\omega(t)$. We assume a zero risk premium $\lambda_\omega = 0$ such that the stochastic discount factor is still in line with equation (3.4) and optimal consumption does not change and is in line with equation (3.15). We still use the martingale method in a similar way as before, as described in Cox and Huang (1989), to find the optimal portfolio weights. An alternative choice for λ_ω would bring us to the setting of Hee and Pearson (1991).

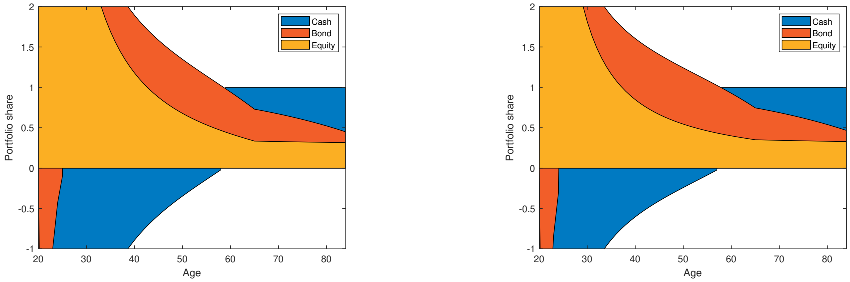
We present the approximation of optimal portfolio weights, where derivations can be found in **Appendix B** and $\tilde{L}(t)$ denotes the value of human capital that excludes the social benefits

$$\bar{\omega}_s(t) = \hat{\omega}_s(t) - \beta_{s,i}\frac{\tilde{L}(t)}{F(t)} \tag{3.28}$$

$$\bar{\omega}_p(t) = \hat{\omega}_p(t). \tag{3.29}$$

The optimal stock allocation $\bar{\omega}_s(t)$ in equation (3.28) contains an additional negative term compared to $\hat{\omega}_s(t)$ in equation (3.19). The intuition of this term is that the optimal stock allocation is lowered in proportion to the exposure coefficient $\beta_{s,i}$, which indicates the exposure of human capital to stock market risk. We exclude the value of the state pension in this correction term since it is not exposed to these shocks and therefore we use $\tilde{L}(t)$ instead of $L(t)$, where $\tilde{L}(t)$ reflects labour income only.

We set the order of magnitude of the volatility of permanent shocks in line with Vestman et al. (2021) to 6 %. We calculate the exposure coefficient $\beta_{s,i}$ by multiplying the permanent income stock correlation by the ratio of the permanent income volatility and stock return volatility. In principle, this is the beta from a CAPM regression with permanent income shocks on the market return. We have defined the volatility of stock returns at 18 %. The correlation coefficients 0, 0.2, 0.4 overlap with the correlation coefficients that Cocco et al. (2005) consider and seem to capture the distribution presented in Bagliano et al. (2021), but are slightly lower than in Campbell and Viceira (2002), leading to estimates for the exposure coefficient $\beta_{s,i}$ as follows: 0, 0.07 and 0.13. These values for $\beta_{s,i}$ lead to a volatility of 0 %, 1.26 % and 2.34 % of the last term in equation (3.27). We present the optimal asset allocation in Figure 3.8 for agents with a flat income profile but differ in their exposure of labour income to the stock market while consumption is optimal. We switch the sign of the exposures if financial wealth is negative such that the reader can interpret the exposures more intuitively. In a median scenario financial wealth is non-negative for a participant with exposure coefficient 0, 0.07 and 0.13 at the age of 20 for all exposure coefficients.



(a) Optimal asset allocation with a flat income profile and $\beta_{s,1} = 0$.

(b) Optimal asset allocation with a flat income profile and $\beta_{s,3} = 0.13$.

Figure 3.8: The figure shows the optimal asset allocation in a median scenario while consumption is optimal for the model of Van Bilsen et al. (2020) extended by a permanent shock consisting of an idiosyncratic component and an aggregate component for a participant with a flat income profile. We only present the optimal asset allocation for the exposure coefficients at the extreme within our framework: $\beta_{s,1} = 0$ and $\beta_{s,3} = 0.13$. The full parameterization is presented in Table 3.1.

Under the assumption of a flat income profile defined in Section 3.3, human capital becomes stock-like and this is increasing for individual i in $\beta_{s,i}$. We expect that this leads to lower stock exposures at all ages. For example, at the age of 45 where the life cycle with $\beta_{s,i} = (0, 0.07, 0.13)$ has an equity exposure in terms of financial wealth equal to $\bar{\omega}_s = (0.8771, 0.7725, 0.6633)$. We present optimal consumption for the different exposure coefficients in Figure 3.9. We see a higher consumption standard for a lower exposure coefficient. This is explained by a convexity term in the value of human capital. Furthermore, we will calculate the welfare loss of a mismatch in the exposure coefficient of labour income to stock market risk β_s for the asset allocation 'one size fits all' pension contract while consumption is optimal. We present the results in Table 3.4.

The welfare losses of a mismatch in β_s can be up to 12.72 % for underestimating the exposure coefficient for the asset allocation while consumption is optimal. This

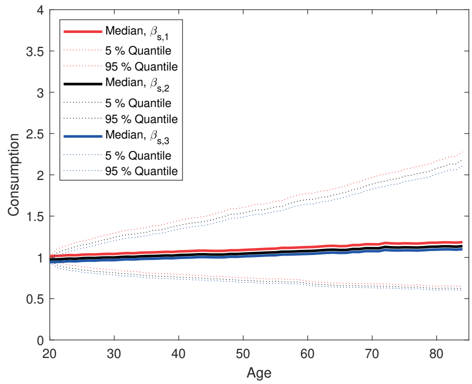


Figure 3.9: The figure shows the consumption distribution for a participant with a flat income profile and exposure coefficients $\beta_{s,1} = 0$, $\beta_{s,2} = 0.07$ and $\beta_{s,3} = 0.13$. The full parameterization is presented in Table 3.1.

Panel A: We set $\sigma_\omega = 0.06$			
	$\beta_{s,1} = 0$	$\beta_{s,2} = 0.07$	$\beta_{s,3} = 0.13$
$\beta_{s,1} = 0$	0 %	-2.86 %	-12.72 %
$\beta_{s,2} = 0.07$	0.09 %	0 %	-1.06 %
$\beta_{s,3} = 0.13$	-1.22 %	-0.15 %	0 %
Panel B: We set $\sigma_\omega = 0$			
	$\beta_{s,1} = 0$	$\beta_{s,2} = 0.07$	$\beta_{s,3} = 0.13$
$\beta_{s,1} = 0$	0 %	-0.79 %	-3.40 %
$\beta_{s,2} = 0.07$	-0.79 %	0 %	-0.42 %
$\beta_{s,3} = 0.13$	-2.25 %	-0.52 %	0 %

Table 3.4: Welfare losses from mismatch in exposure coefficient of labour income to stock market risk β_s. We calculate these welfare losses for an individual with a flat income profile. The financial market and individual parameters are described in Table 3.1. In the columns, we define the exposure coefficient and in the rows the exposure coefficient used as inadequate investment strategy in the 'one size fits all' pension contract while consumption is optimal. We do not adjust the volatility of the last term in equation (3.27) when $\sigma_\omega = 0\%$.

welfare loss is explained by inserting a lower value for β_s in the second term in equation (3.28) than optimal. This means that we underestimate the correlation of human capital with the stock market. In this case, the economic impact of underestimating the exposure coefficient is that excess equity risk is taken. For example, when the true exposure coefficient is 0.13, but the 'one size fits all' pension contract assumes an exposure coefficient of 0, the welfare loss is highest —due to excess equity risk— at 12.72 %. We see that the welfare loss shrinks to 3.40 % if we set σ_ω in equation (3.28) equal to 0 %.

The welfare losses of a mismatch in β_s can be up to 2.25 % for overestimating the exposure coefficient for the asset allocation while consumption is optimal. This welfare loss is explained by inserting a higher value for β_s in the second term in equation (3.28) than optimal. This means that we overestimate the correlation of human capital with the stock market. In this case, the economic impact of overestimating the exposure coefficient is that too low equity risk is taken. For example, when the true exposure coefficient is 0, but the 'one size fits all' pension contract assumes an exposure coefficient of 0.13 the welfare loss is highest —due to too low equity risk— at 1.22 %. We see that the welfare losses can increase to a maximum of 2.25 % if we set σ_ω in equation (3.28) equal to 0 %. We also find a very small welfare gain of 0.09 % when we assume an exposure coefficient of 0.07 instead of the true exposure coefficient of 0 for the asset allocation in the setting with $\sigma_\omega = 0.06$. The economic explanation is that we have an incomplete market setting in which the investor cannot hedge against permanent idiosyncratic income shocks, but we do not take this volatility into account in solving the optimal asset allocation.

A directly related paper is by Bagliano et al. (2014), who calculate welfare losses for individuals with permanent income shocks correlated to the stock market. They have a similar asset menu as in our paper and consider as suboptimal asset allocation strategies the 'age rule','Target Date Fund rule' and '1/N rule'. They consider a correlation coefficient up to 0.2 —where we also consider 0.4— and for normal labour income variance they find welfare losses up to 2 %. Some differences with our setting is that they allow for transitory shocks, a bequest motive, a state pension where the level is uncertain until retirement age, and portfolio and consumption constraints.

A broader related paper is by Bagliano et al. (2021), who calculate welfare losses for agents with human capital in the form of facing the possibility of long term unemployment. They find that a rule of thumb asset allocation strategy, such as the 'age rule' or 'Target Date Fund rule', lead to welfare losses in the order of 3 % - 9 % for an agent who faces the possibility of long term unemployment. The economic intuition is mainly excess equity risk at the start of the life cycle and too little equity exposure in retirement. Branger et al. (2014) allow for unemployment risk that varies with age and the business life cycle and find that neglecting this can lead to a welfare loss of 2.4 %. The economic intuition for this welfare loss is that the standard life cycle takes excess risk since it ignores the riskiness of human wealth.

We have also analyzed whether we can incorporate the interest rate level in the expected income growth rate next to a career path as in Munk and Sørensen (2010). They argue that wage increases occur more often in a high interest rate environment than in a low interest rate environment. However, this extension was economically less intuitive to us and therefore we choose to omit this. Changes in interest rates already have an effect on the value of human capital via the discounting mechanism.

The welfare losses that we find in Section 3.5 are larger than those reported in the literature, see for example Van Ewijk et al. (2017). Moreover that paper argues that the saving decision is a more important determinant of welfare than the investment strategy. For the case of a suboptimal saving decision they find welfare losses no larger than 5 %. However, the largest welfare loss they report is slightly larger than 10 % for a portfolio of stocks that is not well diversified. Although we consider the impact of inadequate portfolio decisions only, and assume the optimal saving decision, we still find sizable welfare losses in this section. This case of an inadequate ratio of human wealth over financial wealth due to different career paths is not considered by Van Ewijk et al. (2017). Our results can be explained by the fact that without restrictions the exposures can be very extreme in this case, leading to large welfare losses.

3.6 Inadequate pension contract with constraints

In Section 3.6.1 we analyze the optimal pension contract with constraints. In particular, we calculate welfare losses of imposing constraints in the pension contract of 'no constraints versus strict constraints (NC vs SC)' and 'strict constraints versus mild constraints (SC vs MC)'. This analysis is related to the investment rules of the pension contract set by the authorities and not to the 'one size fits all' pension contract. In Section 3.6.2 we calculate the welfare losses of the use of inadequate preference parameters in the 'one size fits all' pension contract if short positions are not allowed.

3.6.1 Welfare losses of different restrictions in the contract

We determine optimal portfolio weights in a pension contract with constraints. We consider the case (SC) in which the portfolio weights of all asset classes are between 0 and 1 as in Van Bilsen et al. (2020). This is for stocks $\omega_s(t)$, bonds $\omega_p(t)$ and cash $\omega_c(t)$ defined as follows

$$0 \leq \omega_s(t) \leq 1 \tag{3.30}$$

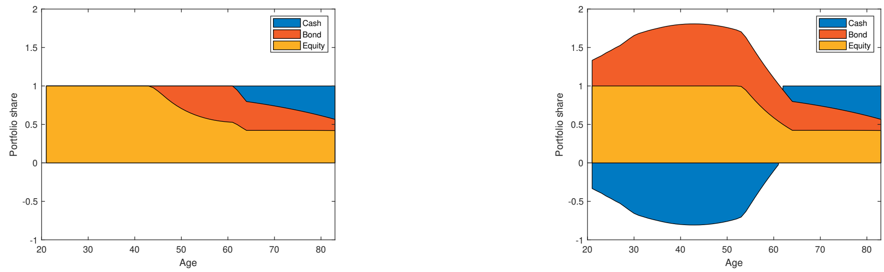
$$0 \leq \omega_p(t) \leq 1 \tag{3.31}$$

$$0 \leq \omega_c(t) \leq 1. \tag{3.32}$$

Portfolio weights should also sum up to 1 as follows

$$\omega_s(t) + \omega_p(t) + \omega_c(t) = 1. \tag{3.33}$$

The life cycle model developed by Merton (1969) shows that younger participants typically have optimal allocations to stocks of $\omega_s(t) \geq 1$ such that equation (3.30) is binding. We allow participants to hedge the interest rate risk of accumulated pension wealth and do this by removing the constraint in equation (3.32). Therefore, the participant can —on top of full allocation to equities $\omega_s(t) = 1$— allocate the amount of accumulated financial wealth to bonds financed by a short position in cash. We refer to this as the setting with (MC). Although pension providers use swaps to hedge interest rate risk, our strategy is similar, but an attractive modelling approach such that we can keep the current asset menu. We write the algorithm for the numerical programming on annual frequency in line with Van Bilsen et al. (2020) and Koijen et al. (2010). We present the details of our numerical solution technique at an annual frequency in **Appendix C**.[9] We present the optimal asset allocation in the setting with SC and MC in Figure 3.10.



(a) Optimal asset allocation of financial wealth with SC.

(b) Optimal asset allocation of financial wealth with MC.

Figure 3.10: The figure shows the optimal asset allocation in a median scenario while consumption is optimal for the model of Van Bilsen et al. (2020) with constraints. The full parameterization is presented in Table 3.1.

We see that for the setting with SC the agent is fully invested in stocks until the age of 44. In the setting with MC we see that the possibility of a short position in cash leads to the fact that the agent invests already some amount in long term bonds at an earlier stage in life. We also present the consumption distribution for SC and MC in Figure 3.11, where savings are non-negative. We see that agents have small savings in the first years of their career. This is because of the restriction that consumption cannot exceed accumulated financial wealth (labour income plus contributions multiplied by investment returns). We also have less extreme

[9]We have that the optimization problem we solve in this section is identical to equation (3.11) apart from (i) a small difference in the timing of matching the duration of the wealth consumption ratio and (ii) a discretization grid of annual decision making versus approximately twice a week (i.e. $dt = \frac{1}{100}$ and (iii) more economic scenarios due to less frequent decision making (50,000 versus 2,000).

investment returns due to restrictions on the asset allocation (see equations (3.30)-(3.33)). We see that optimal median consumption is mainly upward sloping. We visually see marginal differences between optimal consumption under SC and MC. When we compare the consumption pattern in Figure 3.11 to the consumption pattern without constraints in Figure 3.4 we see a lower consumption standard if we add constraints to the model. This is explained by lower expected returns since we have borrowing constraints.

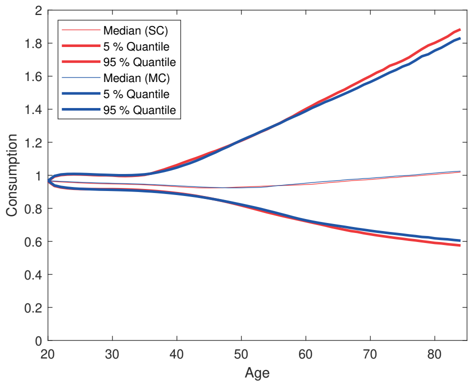


Figure 3.11: The figure shows the consumption distribution for the model of Van Bilsen et al. (2020) with constraints. The full parameterization is presented in Table 3.1.

A related paper is by Bovenberg et al. (2007), who have estimated the welfare losses of constraints in the pension contract to 2.8 % in a setting without interest rate risk. We calculate that imposing SC in the pension contract leads to a welfare loss of 5.45 % (NC vs SC). We find a higher welfare loss than Bovenberg et al. (2007) when we take interest rate risk into account. This welfare loss is just because more than 100 % equities is not allowed. When we impose MC this leads to a welfare loss of only 4.81 %. We explain this smaller welfare loss of 0.68 % by slightly relaxing the constraints due to the removing of equation (3.32) such that a short position in cash was allowed (SC vs MC). Therefore, we conclude that a limited short position to partially hedge interest rate risk for accumulated pension wealth is marginally attractive. In Section 3.6.2 we focus on the setting with SC.

3.6.2 Welfare losses of a 'one size fits all' contract with inadequate levels of risk aversion

We calculate the welfare loss of a mismatch in the level of risk aversion and time preference used for both the investment and consumption strategy in a 'one size fits all' pension contract, but in a setting with (SC). When we compare the wel-

Panel A: Welfare losses				
	SC		NC	
	$\delta = 0.02$	$\delta = 0.03$	$\delta = 0.02$	$\delta = 0.03$
$\gamma = 3$	-1.16 %	0.27 %	-3.24 %	-3.32 %
$\gamma = 5$	-1.07 %	0 %	-0.33 %	0 %
$\gamma = 10$	-7.44 %	-5.49 %	-9.46 %	-7.96 %
Panel B: Equity allocations[10]				
	SC		NC	
	Age to 100 % exposure		Optimal exposure	
	$\delta = 0.02$	$\delta = 0.03$	$\delta = 0.02$	$\delta = 0.03$
$\gamma = 3$	50	54	37.04 %	37.04 %
$\gamma = 5$	41	44	22.22 %	22.22 %
$\gamma = 10$	27	30	11.11 %	11.11 %
Panel C: Optimal wealth-consumption ratio				
	SC		NC	
	$\delta = 0.02$	$\delta = 0.03$	$\delta = 0.02$	$\delta = 0.03$
$\gamma = 3$	35.14	32.82	32.30	29.82
$\gamma = 5$	35.51	33.97	33.57	31.95
$\gamma = 10$	36.39	35.49	35.10	34.21

Table 3.5: Welfare losses from a pension contract which is optimized for an individual with different risk aversion γ used for both the investment and consumption strategy for a participant with a flat career path under SC (Panel A). We have also included the age until which a full allocation to equities is optimal in a median scenario (Panel B) and optimal wealth-consumption ratio for different values of risk aversion that we use for inadequate investment and consumption strategies in the first year (Panel C). The financial market parameters are described in Table 3.1. The imposed parameters for the 'one size fits all' pension contract are $\gamma = 5$ and $\delta = 0.03$. We also report the numbers from Table 3.2.

fare losses of a 'one size fits all' pension contract with risk aversion presented in

[10]Panel B, SC: Age until full allocation to equities is optimal. Panel B, NC: Optimal equity exposure in terms of total wealth. The optimal equity exposure as a fraction of total wealth in equation (3.13) depends on the risk aversion, volatility of stock returns and equity risk premium.

Table 3.5 to the results in Table 3.2, we find lower welfare losses under (SC). The economic intuition is that optimal consumption and asset allocation are less extreme under SC such that the 'one size fits all' pension contract is closer to the optimal solution than in the setting with (NC) since we impose borrowing constraints. Firstly, the optimal equity exposure of 22.22 % in terms of total wealth is no longer reached since a leveraged position is not allowed. Secondly, the wealth-consumption is higher if short positions are not allowed (33.97 versus 31.95) leading to a lower optimal consumption standard. A higher consumption standard can no longer be financed by extreme asset allocations. Obviously, the stochastic consumption stream with $\hat{\gamma} = 5$ and $\hat{\delta} = 0.03$ imposed in the pension contract still yields a zero welfare loss because the true preference parameters $\gamma = 5$ and $\delta = 0.03$ are used.

For example, the welfare loss of a stochastic consumption stream with $\hat{\gamma} = 5$ and $\hat{\delta} = 0.03$ imposed in the pension contract while true preference parameters are $\gamma = 10$ and $\delta = 0.03$ is equal to 5.49 %. The economic reason for this loss —if short positions are not allowed— is excessive risk taking. Obviously too much risk taking in terms of financial wealth is still the case, but less pronounced, since the optimal median equity exposure for $\gamma = 10$ is 100 % for an agent younger than 30, while a 100% equity exposure is optimal for an agent younger than 44 for $\gamma = 5$.

For example, the welfare loss of a stochastic consumption stream with $\hat{\gamma} = 5$ and $\hat{\delta} = 0.03$ imposed in the pension contract while true preference parameters are $\gamma = 5$ and $\delta = 0.02$ is equal to 1.07 %. The economic reason for this loss is a too high consumption standard or equivalently too low savings, in the younger years. Obviously, it cannot be true that a suboptimal pension contract leads to a welfare gain. Nevertheless, a welfare gain of 0.27 % is reported in Table 3.5 for the individual with true preference parameters $\gamma = 3$ and $\delta = 0.03$ while the 'one size fits all' pension contract assumes $\hat{\gamma} = 5$ and $\hat{\delta} = 0.03$. We provide a justification for this in **Appendix D**. The key reason for the approximation error is that we assume annual decision making instead of decision making in continuous time.

It does not depend on the time preference parameter as is discussed in Section 3.3. Therefore, the optimal asset allocation for column $\delta = 0.02$ and column $\delta = 0.03$ is identical.

3.7 Summary and conclusion

We find that welfare losses of a ‘one size fits all’ pension contract are potentially large for agents in a setting with interest rate risk if short positions are allowed. We find that the worst welfare loss within our framework due to an inadequate level of risk aversion is 9.64 % for an individual with a flat career path (NC).[11] We find within our framework a welfare loss in a similar order of magnitude for an inadequate age dependent lifecycle (NC). We extend the model for the representative agent by a career path and find a significant welfare loss when the ‘one size fits all’ pension contract assumes a different career path for the asset allocation while consumption remains optimal (NC). For the particular agents and calibration, we find a maximum welfare loss larger than 20 %. We then allow for permanent income shocks with equity exposure and find that the worst welfare loss within our framework of an inadequate exposure coefficient in the ‘one size fits all’ pension contract used for the asset allocation while consumption is optimal is 12.72 % (NC).

We conclude that a pension contract with mild constraints is attractive for our calibration in a setting with interest rate risk. We find that imposing strict portfolio constraints in a pension contract in which all asset allocations are between zero and one compared to a pension contract with no restrictions leads to a welfare loss of 5.45 % (NC vs SC). We find that imposing mild constraints in the pension contract in which the individual can hedge interest rate risk while having a full allocation to equities compared to a pension contract with no restrictions lead to a welfare loss of 4.81 %. Therefore, we create a welfare gain of 0.68 % by a limited short position in cash (SC vs MC). In the pension contract, if short positions are not allowed, the maximum welfare loss of inadequate preference parameters shrinks to 7.44 % (SC).

Our paper has a few limitations. We mainly focus our analysis on the setting where short positions are allowed and contributions are stochastic which is dif-

[11] Recall that we have defined the following abbreviations: NC = no constraints; SC = strict constraints; MC = mild constraints.

ferent from the pension industry. Therefore, we provide some guidelines on what characteristics of the participant the pension industry could take into account in further research in setting the pension contract.

A first extension would be to analyze welfare losses of a 'one size fits all' pension contract, if short positions are not allowed, for an inadequate age dependent investment strategy and for agents with different assumptions on human capital. Furthermore, it might be interesting to identify the best 'one size fits all' pension contract for a heterogeneous population which we do not address in this paper.

3.8 Appendix

Appendix A is a replication of Van Bilsen et al. (2020). Appendix B considers derivations for the optimal asset allocation for the reference agent with different assumptions on labour income. Appendix C describes the numerical solution technique. Appendix D presents some robustness checks on the numerical results, but also elaborates on the impact of the decision frequency.

A. Replication Van Bilsen et al. (2020)

A1. Optimal asset allocation without human wealth

We start by replicating the optimal portfolio weights in terms of total wealth. We can formulate the Lagrange function L as follows with Lagrange multiplier y

$$L = E\left[\int_0^T \exp\{-\delta t\} \cdot \frac{1}{1-\gamma} c(t)^{1-\gamma} dt\right] + y\left(W(0) - E\left[\int_0^T M(t)c(t)dt\right]\right) \tag{3.34}$$

$$= \int_0^T E\left[\exp\{-\delta t\} \cdot \frac{1}{1-\gamma} c(t)^{1-\gamma} - yM(t)c(t)\right] dt + yW(0) \tag{3.35}$$

where the stochastic discount factor $M(t)$ is defined as follows

$$M(t) = \exp\Big\{ - \int_0^t (r(t) + \frac{1}{2}\phi'\phi)dt + \phi' \int_0^t dZ(t)\Big\} \tag{3.36}$$

$$\begin{aligned} r(t) &= r(0) \cdot \exp\{-\kappa_r \cdot t\} + \overline{r}(1 - \exp\{-\kappa_r \cdot t\}) \\ &\quad + \sigma_r \exp\{-\kappa_r \cdot t\} \int_0^t \exp\{\kappa_r \cdot s\} dZ_r(s). \end{aligned} \tag{3.37}$$

We calculate the derivative of the inner part of the expectation with respect to $c(t)$ as follows and determine the optimal consumption $c^*(t)$

$$\begin{aligned} \exp\{-\delta t\} \cdot c^*(t)^{-\gamma} &= yM(t) \\ c^*(t) &= \Big(\exp\{\delta t\} \cdot yM(t)\Big)^{-\frac{1}{\gamma}}. \end{aligned} \tag{3.38}$$

The Lagrange multiplier y is defined such that the budget constraint holds at the optimum

$$y = \left(\frac{W(0)}{\underbrace{E\Big[\int_0^T M(t)^{1-\frac{1}{\gamma}} \exp\{\delta t\}^{-\frac{1}{\gamma}} dt\Big]}_{A(0)}} \right)^{-\gamma}. \tag{3.39}$$

Munk (2017) shows that we can formulate optimal consumption $c^*(t)$ also as a function of $W(t)$ instead of $W(0)$ only with $A^*(t)$ the wealth consumption ratio introduced in the next formula

$$c^*(t) = \frac{W(t)}{A^*(t)}. \tag{3.40}$$

We denote the wealth consumption ratio $A^*(t)$ as follows

$$\begin{aligned} A^*(t) &= \int_0^{T-t} E_t\Big[\frac{M(t+h)}{M(t)} \frac{c^*(t+h)}{c^*(t)}\Big] dh \\ &= \int_0^{T-t} \exp\{-d^*(t,h)h\} dh \end{aligned} \tag{3.41}$$

where we derive $d^*(t,h)$ as follows

$$
\begin{aligned}
d^*(t,h) &= -\frac{1}{h}\log\left(E_t\left[\frac{M(t+h)}{M(t)}\frac{c^*(t+h)}{c^*(t)}\right]\right) \\
&= \frac{1}{h}\left[\left(1-\frac{1}{\gamma}\right)\int_0^h\left(r(t)+\kappa_r B_r(v)(\overline{r}-r(t))+\frac{1}{2}\phi'\phi\right)dv\right. \\
&\quad \left.-\frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2\int_0^h(\phi_r-B_r(v)\sigma_r)^2dv\right]-\frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2\phi_s^2+\frac{\delta}{\gamma}. \qquad (3.42)
\end{aligned}
$$

We define the optimal duration of the wealth consumption ratio $D_A(t)$ as follows

$$D_A(t) = (1-\frac{1}{\gamma})\int_0^{T-t}\frac{V^*(t,h)}{V^*(t)}B_r(h)dh \qquad (3.43)$$

$$V^*(t) = c^*(t)\cdot\int_0^{T-t}\exp\{-d^*(t,h)h\}dh \qquad (3.44)$$

$$V^*(t,h) = c^*(t)\cdot\exp\{-d^*(t,h)h\}. \qquad (3.45)$$

The dynamics of the market-consistent value of consumption $\text{dlog}(V^*(t))$ and the dynamics of log wealth $\text{dlog}(W(t))$ are as follows

$$\text{dlog}(V^*(t)) = (...)dt-\left(\frac{1}{\gamma}\phi_r+D_A(t)\sigma_r\right)dZ_r(t)-\frac{1}{\gamma}\phi_s dZ_s(t) \qquad (3.46)$$

$$\text{dlog}(W(t)) = (...)dt-\omega_p(t)B_r(h)\sigma_r dZ_r(t)+\omega_s(t)\sigma_s dZ_s(t). \qquad (3.47)$$

We get the optimal asset allocation by solving the following equations for $\omega_p^*(t)$ and $\omega_s^*(t)$, which we obtain by equating the terms in front of the brownian motions for $\text{dlog}(V^*(t))$ and $\text{dlog}(W(t))$

$$\omega_s(t)\sigma_s = -\frac{1}{\gamma}\phi_s \qquad (3.48)$$

$$-\omega_p(t)B_r(h)\sigma_r = -\left(\frac{1}{\gamma}\phi_r+D_A(t)\sigma_r\right). \qquad (3.49)$$

The solution is

$$\omega_s^*(t) = -\frac{1}{\gamma}\frac{\phi_s}{\sigma_s} \tag{3.50}$$

$$\omega_p^*(t) = \frac{1}{\gamma}\frac{\phi_r}{B_r(h)\sigma_r} + \frac{D_A(t)}{B_r(h)}. \tag{3.51}$$

A2. Optimal asset allocation with human wealth

We define the value of the contribution to human capital h periods from time t by $L(t,h)$ as follows

$$L(t,h) = E_t\left[\frac{M(t+h)}{M(t)}O(t+h)\right] \tag{3.52}$$

and taking the integral over h defines $L(t)$, the value of human capital at time t.

$$L(t) = \int_0^{T-t} L(t,h)dh \tag{3.53}$$

We can simplify the formula for $L(t,h)$ as follows

$$\begin{aligned} L(t,h) &= O(t+h)\cdot E_t\left[\frac{M(t+h)}{M(t)}\right] \\ &= O(t+h)\cdot \exp\left\{-r(t)B_r(h) - m(h)\right\} \end{aligned} \tag{3.54}$$

with $m(h)$ defined as follows.

$$m(h) = \left(\overline{r} - \frac{\lambda_r\sigma_r}{\kappa_r} - \frac{1}{2}\frac{\sigma_r^2}{\kappa_r^2}\right)(h - B_r(h)) + \frac{1}{4\kappa_r}B_r^2(h)\sigma_r^2 \tag{3.55}$$

Alternatively, we can simplify the formula for $L(t,h)$ as follows

$$L(t,h) = O(t+h)\cdot \exp\left\{-\int_0^h R(t,v)dv\right\} \tag{3.56}$$

with the instantaneous nominal forward interest rate $R(t, v)$.

$$R(t, v) = E_t[r(t + v)] - \lambda_r \sigma_r B_r(v) - \frac{1}{2} B_r^2(v) \sigma_r^2 \tag{3.57}$$

We calculate the duration of human capital $D_L(t)$ as follows

$$D_L(t) = \int_0^{T-t} \frac{L(t, h)}{L(t)} B_r(h) dh. \tag{3.58}$$

We derive the dynamics of human wealth $dL(t)$ as follows

$$dL(t) = \Big(r(t) - \lambda_r \sigma_r D_L(t) \Big) L(t) dt - D_L(t) \sigma_r L(t) dZ_r(t) - O(t) dt. \tag{3.59}$$

The dynamics of total wealth $dW(t)$ as the sum of human wealth $dL(t)$ and financial wealth $dF(t)$

$$\begin{aligned} dW(t) &= dL(t) + dF(t) \qquad (3.60) \\ &= (...)dt + \hat{\omega}_s(t) \sigma_s \frac{F(t)}{W(t)} W(t) dZ_s(t) - \Big(\hat{\omega}_p(t) B_r(h) \frac{F(t)}{W(t)} + D_L(t) \frac{L(t)}{W(t)} \Big). \end{aligned}$$

We solve the following equations, which we obtain by equating the stochastic terms of dlog$(W(t))$ and dlog$(V^*(t))$, to obtain the optimal asset allocation

$$\hat{\omega}_s(t) \sigma_s \frac{F(t)}{W(t)} = -\frac{1}{\gamma} \phi_s \tag{3.61}$$

$$-\Big(\hat{\omega}_p(t) B_r(h) \frac{F(t)}{W(t)} + D_L(t) \frac{L(t)}{W(t)} \Big) \sigma_r = -\Big(\frac{1}{\gamma} \phi_r + D_A(t) \sigma_r \Big). \tag{3.62}$$

We derive the optimal portfolio asset allocation in terms of financial wealth for stocks $\hat{\omega}_s(t)$ and for bonds $\hat{\omega}_p(t)$ as follows

$$\hat{\omega}_s(t) = \frac{W(t)}{F(t)}\omega_s^*(t) \tag{3.63}$$

$$\hat{\omega}_p(t) = \frac{W(t)}{F(t)}\omega_p^*(t) - \frac{L(t)}{F(t)}\frac{D_L(t)}{B_r(h)}. \tag{3.64}$$

B. Optimal asset allocation with different assumptions on human capital

We calculate the contribution to human capital h periods from time t —correlated with the stock market— $\tilde{L}(t,h)$ by discounting the labour income defined in equation (3.27) as follows

$$\begin{aligned}
\tilde{L}(t,h) &= E_t\left[\frac{M(t+h)}{M(t)}\overline{O}_i(t+h)\right] \\
&= E_t\left[\frac{M(t+h)}{M(t)} \cdot \tilde{O}_i(t+h) \cdot \exp\left\{\sigma_\omega \int_0^{t+h} dZ_\omega(s) + \beta_{s,i}\sigma_s \int_0^{t+h} dZ_s(s)\right\}\right] \\
&= \tilde{O}_i(t+h) \cdot \exp\left\{\sigma_\omega \int_0^{t} dZ_\omega(s) + \beta_{s,i}\sigma_s \int_0^{t} dZ_s(s)\right\} \\
&\cdot E_t\left[\exp\left\{\sigma_\omega \int_0^{h} dZ_\omega(t+s)\right\}\right] \cdot E_t\left[\frac{M(t+h)}{M(t)} \cdot \exp\left\{\beta_{s,i}\sigma_s \int_0^{h} dZ_s(t+s)\right\}\right].
\end{aligned} \tag{3.65}$$

We calculate the first expectation of equation (3.65), which only has a convexity term since the idiosyncratic permanent shock has mean zero, as follows

$$\begin{aligned}
&E_t\left[\exp\left\{\sigma_\omega \int_0^{h} dZ_\omega(t+s)\right\}\right] \\
&= \exp\left\{\frac{1}{2}\sigma_\omega^2 h\right\}.
\end{aligned} \tag{3.66}$$

We then focus on the second expectation of equation (3.65), which contains the exposure to stock market risk, as follows

$$\begin{aligned}
&E_t\Bigg[\frac{M(t+h)}{M(t)}\cdot\exp\Big\{\beta_{s,i}\sigma_s\int_0^h dZ_s(t+s)\Big\}\Bigg]\\
&=E_t\Bigg[\exp\Big\{-\int_0^h\Big(r(t+s)+\frac{1}{2}\phi'\phi\Big)ds+\phi'\int_0^h dZ(t+s)\Big\}\\
&\cdot\exp\Big\{\beta_{s,i}\sigma_s\int_0^h dZ_s(t+s)\Big\}\Bigg]\\
&=E_t\Bigg[\exp\Big\{-\int_0^h\Big(r(t+s)+\frac{1}{2}\phi'\phi\Big)ds+\phi'\int_0^h dZ(t+s)+\beta_{s,i}\sigma_s\int_0^h dZ_s(t+s)\Big\}\Bigg]\\
&=E_t\Bigg[\exp\Big\{-\int_0^h\Big(r(t+s)+\frac{1}{2}\phi'\phi\Big)ds+\phi_r\int_0^h dZ_r(t+s)\\
&+(\phi_s+\beta_{s,i}\sigma_s)\int_0^h dZ_s(t+s)\Big\}\Bigg]\\
&=E_t\Bigg[\exp\Big\{-\int_0^h\Big(r(t+s)+\frac{1}{2}\phi'\phi\Big)ds+\phi_r\int_0^h dZ_r(t+s)\Big\}\Bigg]\\
&\cdot E_t\Bigg[\exp\Big\{(\phi_s+\beta_{s,i}\sigma_s)\int_0^h dZ_s(t+s)\Big\}\Bigg].
\end{aligned}\tag{3.67}$$

The derivation of the first expectation from equation (3.67) is available in Van Bilsen et al. (2020) and the second expectation from equation (3.67) is derived below

$$\begin{aligned}
&E_t\Bigg[\exp\Big\{(\phi_s+\beta_{s,i}\sigma_s)\int_0^h dZ_s(t+s)\Big\}\Bigg]\\
&=\exp\Big\{\int_0^h\Big(\frac{1}{2}\phi_s^2-\lambda_s\beta_{s,i}\sigma_s+\frac{1}{2}\beta_{s,i}^2\sigma_s^2\Big)dv\Big\}\\
&=\exp\Big\{\frac{1}{2}\phi_s^2h-\lambda_s\beta_{s,i}\sigma_sh+\frac{1}{2}\beta_{s,i}^2\sigma_s^2h\Big\}.
\end{aligned}\tag{3.68}$$

Combining everything we get the following for $\tilde{L}(t,h)$

$$\tilde{L}(t,h) = \tilde{O}_i(t+h) \cdot \exp\Big\{ - r(t)B_r(h) - m(h) \Big\} \cdot \exp\Big\{ \sigma_\omega \int_0^t dZ_\omega(s) + \beta_{s,i}\sigma_s \int_0^t dZ_s(s) + \frac{1}{2}\sigma_\omega^2 \cdot h - \lambda_s \beta_{s,i}\sigma_s h + \frac{1}{2}\beta_{s,i}^2 \sigma_s^2 h \Big\}. \tag{3.69}$$

Alternatively, we define $\tilde{L}(t,h)$ as follows

$$\tilde{L}(t,h) = \tilde{O}_i(t+h) \cdot \exp\Big\{ - \int_0^h R(t,v)dv \Big\} \cdot \exp\Big\{ \sigma_\omega \int_0^t dZ_\omega(s) + \beta_{s,i}\sigma_s \int_0^t dZ_s(s) + \frac{1}{2}\sigma_\omega^2 h - \lambda_s \beta_{s,i}\sigma_s h + \frac{1}{2}\beta_{s,i}^2 \sigma_s^2 h \Big\}. \tag{3.70}$$

In this setting, the optimal asset allocation is no longer in line with equations (3.19) and (3.20). We need to derive the dynamics of human wealth such that we can determine the optimal asset allocation. We will do this step by step and first return to the setting of Van Bilsen et al. (2020)

$$\frac{dL(t,h)}{L(t,h)} = \frac{\partial L(t,h)}{\partial t} dt + \frac{\partial L(t,h)}{\partial r(t)} dr(t) + \frac{1}{2} \frac{\partial^2 L(t,h)}{\partial r^2(t)} d[r(t), r(t)]. \tag{3.71}$$

Since we are interested in the stochastic component of human wealth we can focus on the second term in equation (3.71) since the first and third term will end up as dt terms. We first determine the partial derivative of $L(t,h)$ with respect to $r(t)$

$$\begin{aligned} \frac{\partial L(t,h)}{\partial r(t)} &= \tilde{O}_i(t+h) \cdot \exp\Big\{ - r(t)B_r(h) - m(h) \Big\} \cdot -B_r(h) \\ &= -L(t,h)B_r(h) \end{aligned} \tag{3.72}$$

such that we can calculate the stochastic component as follows.

$$\frac{\partial L(t,h)}{\partial r(t)} dr(t) = (...)dt - L(t,h)B_r(h)\sigma_r dZ_r(t). \tag{3.73}$$

Since we are interested in the dynamics of $L(t)$, we integrate over h and divide and multiply by the value of human capital at time t to get the change

$$\begin{aligned} dL(t) &= (...)dt - \int_0^{T-t} \frac{L(t,h)B_r(h)\sigma_r}{L(t)} L(t)dhdZ_r(t) \\ &= (...)dt - D_L(t)\sigma_r L(t)dZ_r(t). \end{aligned} \tag{3.74}$$

In the setting with permanent shocks to labour income there are some extra terms in the stochastic dynamics of human wealth $\tilde{L}(t)$. We can easily derive from equation (3.69) that the additional term with respect to $dZ_s(t)$ is as follows

$$\begin{aligned} &\beta_{s,i}\sigma_s \int \frac{\tilde{L}(t,h)}{\tilde{L}(t)} \tilde{L}(t)dhdZ_s(t) \\ &= \beta_{s,i}\sigma_s \tilde{L}(t)dZ_s(t). \end{aligned} \tag{3.75}$$

We will neglect the term with respect to $dZ_\omega(t)$ in the analysis since this exposure cannot be hedged in this financial market. We then combine and derive the optimal dynamics of total wealth as follows

$$\begin{aligned} dW(t) &= dL(t) + dF(t) \\ &= (...)dt + \left(\bar{\omega}_s(t)\sigma_s \frac{F(t)}{W(t)} + \beta_{s,i}\sigma_s \frac{\tilde{L}(t)}{W(t)}\right) W(t)dZ_s(t) \\ &- \left(\bar{\omega}_p(t)B_r(h)\frac{F(t)}{W(t)} + D_L(t)\frac{L(t)}{W(t)}\right) \cdot \sigma_r W(t)dZ_r(t). \end{aligned} \tag{3.76}$$

Since dlog($V^*(t)$) does not change in this setting, we equate the stochastic terms of equations (3.46) and (3.76), where straightforward calculations show that the approximated optimal asset allocation to stocks $\bar{\omega}_s(t)$ and bonds $\bar{\omega}_p(t)$ is as follows

$$\bar{\omega}_s(t) = \hat{\omega}_s(t) - \beta_{s,i}\frac{\tilde{L}(t)}{F(t)} \tag{3.77}$$

$$\bar{\omega}_p(t) = \hat{\omega}_p(t). \tag{3.78}$$

We simulate optimal consumption in line with (3.40) to ensure that the participant consumes out of current wealth.

C. Numerical solution technique

We elaborate on the step-by-step approach for the numerical solution technique in line with Van Bilsen et al. (2020). We follow Carroll (2006) for optimal consumption policies. We generate grid points i for interest rates and the fraction of financial wealth. We solve the optimization problem using backward induction as described below for period T, T-1, T-2, ..., 1.

- **Period T** The investor consumes all remaining wealth. Hence, $\frac{c(i,T)}{W_{\text{before}}(i,T)} = 1$ with T the terminal time and $W_{\text{before}}(i,T)$ wealth before consumption at time T.

- **Period T-1** We determine the optimal asset allocation $\omega(t)$ by maximizing $U(i,T)$ for every grid point i for the real interest rate $r(i,T-1)$ with weighting function $w(v)$ with v denoting the node in the Gaussian quadrature[12][13]

$$U(i,T) = \frac{1}{1-\gamma} \sum_{v} \left(R_{\text{tot}}^{\text{proj}}(i,T,v) \right)^{1-\gamma} w(v). \tag{3.79}$$

We project for every grid point i for the real interest rate $r(i,T-1)$, the stock return, cash return and bond return at time T using Gaussian Quadrature $R_s^{\text{proj}}(.) = \exp\{r_s^{\text{proj}}(i,T,v)\}$, $R_p^{\text{proj}}(.) = \exp\{r_p^{\text{proj}}(i,T,v)\}$ and$R_c^{\text{proj}}(.) = \exp\{r_c^{\text{proj}}(i,T,v)\}$. Given a portfolio weight vector, we can, for every grid point i, compute the total return as follows

$$\begin{aligned} R_{\text{tot}}^{\text{proj}}(i,T,v) &= \omega_s(i,T-1) \cdot R_s^{\text{proj}}(i,T,v) + \omega_p(i,T-1) \cdot R_p^{\text{proj}}(i,T,v) \\ &+ \Big(1 - \omega_s(i,T-1) - \omega_p(i,T-1)\Big) \cdot R_c^{\text{proj}}(i,T,v). \end{aligned} \tag{3.80}$$

[12]In MATLAB we use the function *fmincon* and therefore we do not divide by $1-\gamma$.
[13]We base the weighting function and the Gaussian quadrature nodes on the STROUD C++ library.

For every grid point i, we determine the optimal portfolio weights such that $U(i, T)$ is maximized. Optimal policies $\omega_s(i, T-1)$ and $\omega_p(i, T-1)$ and $\max U(i, T-1)$ will be saved.
We observe that optimal portfolio choice at time $T-1$ is independent of $W_{\text{after}}(i, T-1)$ (wealth after consumption at gridpoint i at time $T-1$) because of the CRRA utility function. We determine optimal consumption $c(i, T-1)$ at gridpoint i and time $T-1$. Therefore, we determine the first order condition as follows (see for example equation (20) in the online appendix to Koijen et al, 2010)

$$c(i, T-1) = \left(\sum_v \exp\{-\delta\} \cdot c(i, T)^{-\gamma} R_{\text{tot}}^{\text{proj}}(i, T, v) \cdot w(v) \right)^{-\frac{1}{\gamma}}. \tag{3.81}$$

We have $c(i, T) = W_{\text{after}}(i, T-1) \cdot R_{\text{tot}}^{\text{proj}}(i, T, v)$. Hence,

$$\frac{c(i, T-1)}{W_{\text{after}}(i, T-1)} = \left(\sum_v \exp\{-\delta\} \cdot R_{\text{tot}}^{\text{proj}}(i, T, v)^{-\gamma} R_{\text{tot}}^{\text{proj}}(i, T, v) \cdot w(v) \right)^{-\frac{1}{\gamma}}. \tag{3.82}$$

In order to find $\frac{c(i,T-1)}{W_{\text{before}}(i,T-1)}$ use $W_{\text{after}}(i, T-1) = W_{\text{before}}(i, T-1) - c(i, T-1)$ which implies

$$\frac{c(i, T-1)}{W_{\text{before}}(i, T-1)} = \frac{c(i, T-1)}{W_{\text{after}}(i, T-1) + c(i, T-1)}. \tag{3.83}$$

We compute optimal utility $U(i, T-1)$ at gridpoint i and time $T-1$ as follows (see for example Van Bilsen et al. 2020)

$$\begin{aligned} U(i, T-1) = &\frac{1}{1-\gamma} \left(\frac{c(i, T-1)}{W_{\text{before}}(i, T-1)} \right)^{1-\gamma} \\ &+ \frac{\exp\{-\delta\}}{1-\gamma} \sum_v \left(\left(1 - \frac{c(i, T-1)}{W_{\text{before}}(i, T-1)} \right) R_{\text{tot}}^{\text{proj}}(i, T, v) \right)^{1-\gamma} w(v). \end{aligned} \tag{3.84}$$

We then compute the certainty equivalent $CE(i, T-1)$ for every gridpoint i at time $T-1$ by $\frac{CE(i,T-1)}{W_{\text{before}}(i,T-1)}$ (per dollar wealth before consumption). We do this as follows

$$CE(i, T-1) = \left(\frac{U(i, T-1) \cdot (1-\gamma)}{\exp\{-\delta\}} \right)^{\frac{1}{1-\gamma}}. \tag{3.85}$$

- **Period T-2** For every grid point for the real interest rate $r(i, T-2)$, we project the real interest rate, stock return, cash return and bond return at time $T-1$ using Gaussian Quadrature $r^{\text{proj}}(i, T-1, v)$, $R_s^{\text{proj}}(i, T-1, v)$, $R_p^{\text{proj}}(i, T-1, v)$ and $R_c^{\text{proj}}(i, T-1, v)$ with v denoting the node in the Gaussian quadrature. Given a portfolio weight vector, we can, for every grid point, compute the total return. We then maximize utility, with respect to portfolio choice as follows,

$$\frac{1}{1-\gamma} \sum_v \left(\frac{CE(i, T-1)}{W_{\text{before}}(i, T-1)} \cdot R_{\text{tot}}^{\text{proj}}(i, T-1, v) \right)^{1-\gamma} w(v) \tag{3.86}$$

because,

$$\frac{CE(i, T-1)}{W_{\text{after}}(i, T-2)} = \frac{CE(i, T-1)}{W_{\text{before}}(i, T-1)} \cdot R_{\text{tot}}^{\text{proj}}(i, T-1, v) \tag{3.87}$$

and note $\frac{CE(i,T)}{W_{\text{before}}(i,T)} = 1$ in the last period since the individual consumes all remaining wealth. Hence, we can compute this and we maximize utility with respect to the portfolio strategy. We determine optimal consumption at time $T-2$ in line with equation (3.81) which is the start of a recursive relation. The fraction of wealth which is not consumed at time $T-2$ is multiplied by the certainty equivalent next period and evaluated using the utility function.

- **Period T-3, ..., 1** The strategy for optimal portfolio strategy and optimal consumption for earlier periods work in a similar way.

We then extend the algorithm where we make an explicit distinction between human wealth $L(t)$ and financial wealth $F(t)$. We use forward simulation to determine for each scenario and each point in time the optimal portfolio strategy $\omega(t)$ and optimal consumption $c(t)$ based on the optimal policies defined at the gridpoints. This enables us to determine certainty equivalents of optimal and suboptimal portfolio strategies.

The optimization problem we solve numerically is identical to equation (3.11) apart from a small difference in the timing of matching the duration of the wealth consumption ratio. The solution technique follows Koijen et al. (2010), who also assume a CRRA utility function. We have the budget constraint in equation (3.12).

D. Robustness check on numerical precision

We present (A) the benchmark certainty equivalent of the 'one size fits all' pension contract assuming $\hat{\gamma} = 5$ and $\hat{\delta} = 0.03$ in a setting with restrictions for an individual with true parameters $\gamma = 5$ and $\delta = 0.03$. We then perform a robustness check on the certainty equivalent in a setting with portfolio restrictions by increasing (B) the gridpoints for the fraction of financial wealth and (C) the gridpoints for interest rates. Note that we already use 96 Gauss-Hermite to approximate the standard integrals. We use for the portfolio policy 51 gridpoints for the fraction of financial wealth and interest rates and for consumption policy 1,001 gridpoints for the fraction of financial wealth and interest rates. This is in line with previous literature in defining more gridpoints for consumption policy than for portfolio choice. We double the gridpoints in (B) for the fraction of financial wealth to 102 and 2,002 for portfolio and consumption policy respectively. We double the gridpoints in (C) for interest rates to 102 and 2,002 for portfolio and consumption policy respectively. We report the results in Table 3.6. We find certainty equivalents in a similar order of magnitude for (A)-(C). Recall that labour income is normalized to unity and the agent receives a state pension income of 40 % of labour income.

We present an additional robustness check to the welfare losses of a 'one size fits all' pension contract based on preference parameters in Table 3.7. Panel A repeats the benchmark welfare losses from Table 3.5. Additionally, we are doubling (B) the gridpoints for the fraction of financial wealth and (C) the gridpoints for interest

	(A) Benchmark	(B) Fraction financial wealth	(C) Interest rate
Certainty equivalent	0.9244	0.9244	0.9244
Relative difference	-	$4.55 \cdot 10^{-6}$	$1.29 \cdot 10^{-5}$

Table 3.6: A robustness check to (A) the benchmark certainty equivalent by doubling (B) the gridpoints for the fraction of financial wealth and (C) the gridpoints for interest rates. We assume the 'one size fits all' pension contract that assumes $\hat{\gamma} = 5$ and $\hat{\delta} = 0.03$ for an individual with true parameters $\gamma = 5$ and $\delta = 0.03$. We also report the absolute value of the relative difference which is defined as $|\frac{CE_{\text{new}} - CE_{\text{A}}}{CE_{\text{A}}}|$, where CE_{new} stands for the certainty equivalent in either case B or C and CE_A is the certainty equivalent for the benchmark case (A).

rates respectively. Although we double for this individual (B) the gridpoints for the fraction of financial wealth and (C) the gridpoints for interest rates we still find a small welfare gain for this individual. For the other individuals the welfare losses accross specifications are also in a similar order of magnitude.

The impact of the decision frequency

We have performed several robustness checks (B-C), but we did not consider alternatives on the assumption of annual decision making, for example daily decision making, due to constraints in computer time. This has the consequence that with annual decision making the certainty equivalent on the gridpoints, input to the optimization problem, is approximated less precise. To be precise, we assume in line with Van Bilsen et al. (2020) that 'we impose the assumption that portfolio shares (as shares of total wealth) are always continuously rebalanced between two discrete periods.'[14] This inconsistency leads to an approximation error in the optimal investment and consumption policy. The precision of this approximation of certainty equivalents would increase if we would assume decision making on higher frequency. Especially, we have that the approximation error can accumulate on longer horizons such as a lifecycle optimization problem. This also seems to be the case in Van Bilsen et al. (2020). Therefore, it is the most likely explanation for the approximation error and this seems to explain why we report a welfare gain of using the 'one size fits all' parameters $\hat{\gamma} = 5$ and $\hat{\delta} = 0.03$ for an individual with true parameters $\gamma = 3$ and $\delta = 0.03$ which should of course be a welfare loss if

[14] We impose this assumption such that the algorithm never has to evaluate a realization where next period's wealth is negative.

Panel A: Benchmark welfare loss				
	$\delta = 0.02$		$\delta = 0.03$	
	Welfare loss	Absolute Diff.	Welfare loss	Absolute Diff.
$\gamma = 3$	-1.16 %	-	0.27 %	-
$\gamma = 5$	-1.07 %	-	0 %	-
$\gamma = 10$	-7.44 %	-	-5.49 %	-
Panel B: Fraction financial wealth with welfare loss and absolute difference				
	$\delta = 0.02$		$\delta = 0.03$	
	Welfare loss	Absolute Diff.	Welfare loss	Absolute Diff.
$\gamma = 3$	-1.16 %	0.0010	0.27 %	0.0040
$\gamma = 5$	-1.07 %	0.0003	0 %	-
$\gamma = 10$	-7.45 %	0.0011	-5.50 %	0.0014
Panel C: Interest rate with welfare loss and absolute difference				
	$\delta = 0.02$		$\delta = 0.03$	
	Welfare loss	Absolute Diff.	Welfare loss	Absolute Diff.
$\gamma = 3$	-1.16 %	0.0006	0.27 %	0.0004
$\gamma = 5$	-1.07 %	0.0011	0 %	-
$\gamma = 10$	-7.44 %	0.0005	-5.49 %	0.0005

Table 3.7: Robustness check for welfare losses of a 'one size fits all' pension contract based on preference parameters with SC. We present (A) the benchmark (see Table 3.5) and are doubling (B) the gridpoints for the fraction of financial wealth and (C) the gridpoints for interest rates. We also report the absolute difference with respect to the benchmark welfare effect which is defined as $|\frac{WL_{\text{new}}-WL_{\text{A}}}{WL_{\text{A}}}|$, where WL_{new} stands for the welfare loss in either case B or C and WL_{A} is the welfare loss for the benchmark case (A).

approximation errors in the optimization per grid point would have been avoided. The economic impact of the assumption of annual decision making in the problem we solve numerically is small.[15] A small welfare gain can only be present for an individual whose optimal pension contract is close to the 'one size fits all' pension contract, as is the case for an individual with true parameters $\gamma = 3$ and $\delta = 0.03$ and a pension contract with parameters $\hat{\gamma} = 5$ and $\hat{\delta} = 0.03$.

[15]Note though that although the problem at higher frequency can thus be solved more precisely (apart from the limitations on computer time) the economic rational from decision making at a lower frequency seems much more realistic.